烟台理工学院

Yantai Institute of Technology

（原烟台大学文经学院）

(Wenjing College Yantai University)

机器人学

人工智能学院　杨智勇

二零二一年八月二十日

第4章 机器人的逆向运动学

导读

- 手臂順向運動學 Forward kinematics (FK)

  給予 θ_i（可計算出 ${}^{i-1}_{i}T$），求得 $\{H\}$ 或 ${}^{w}P$

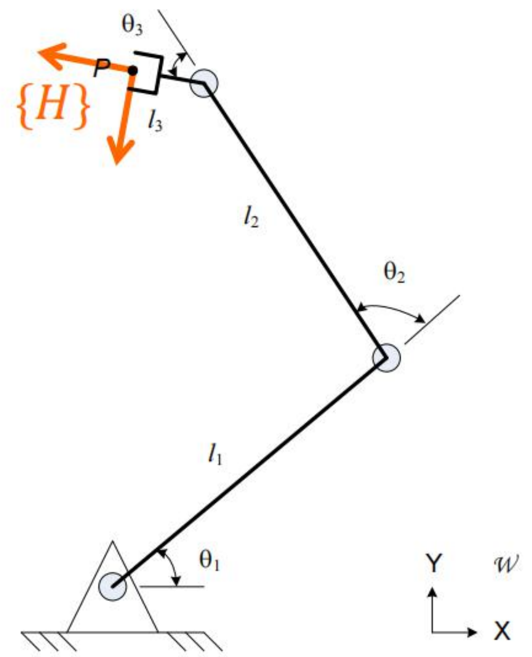
  $${}^{W}_{H}T = f(\theta_1, \dots, \theta_i, \dots, \theta_n)$$

  $${}^{W}P = {}^{W}_{H}T\,{}^{H}P$$

- 手臂逆向運動學 Inverse kinematics (IK)

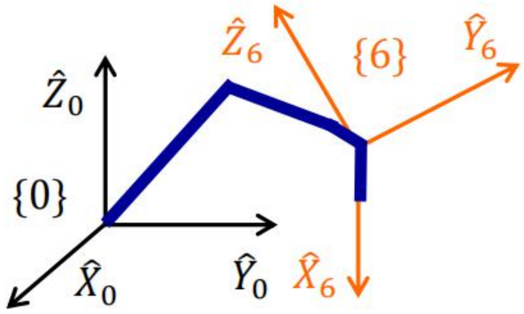
  給予 $\{H\}$ 或 ${}^{w}P$，求得 θ_i

  $$[\theta_1, \dots, \theta_i, \dots, \theta_n] = f^{-1}({}^{W}_{H}T)$$



求解概念

- 假設手臂有6 DOFs
  - 6 個未知的joint angles（θ_i 或 d_i，$i = 1, \ldots, 6$）



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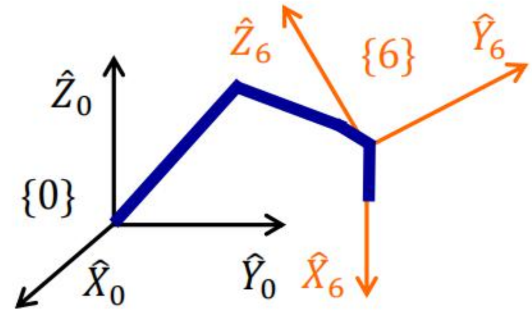

- 在 ${}^W_HT$ 中擷取出含未知數的 ${}^0_6T$，16個數字

$$
{}^0_6T = \left[\begin{array}{ccc|c} & {}^0_6R_{3\times3} & & {}^0P_{6\,org\;3\times1} \\ \hline 0 & 0 & 0 & 1 \end{array}\right]_{4\times4} = \left[\begin{array}{ccc|c} | & | & | & | \\ {}^0\hat{X}_6 & {}^0\hat{Y}_6 & {}^0\hat{Z}_6 & {}^0P_{6\,org} \\ | & | & | & | \\ \hline 0 & 0 & 0 & 1 \end{array}\right]
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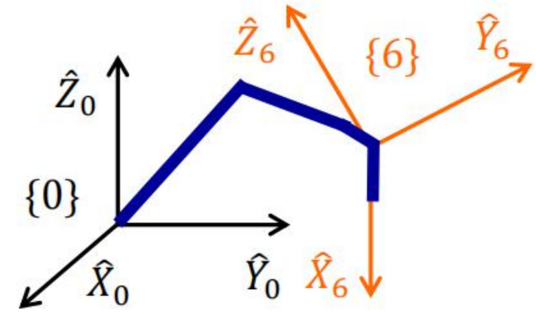

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- 求解
  - 12個nonlinear transcendental equations方程式
  - 6個未知數，6個限制條件

求解概念

- Reachable workspace
  - 手臂可以用一種以上的姿態到達的位置

求解概念

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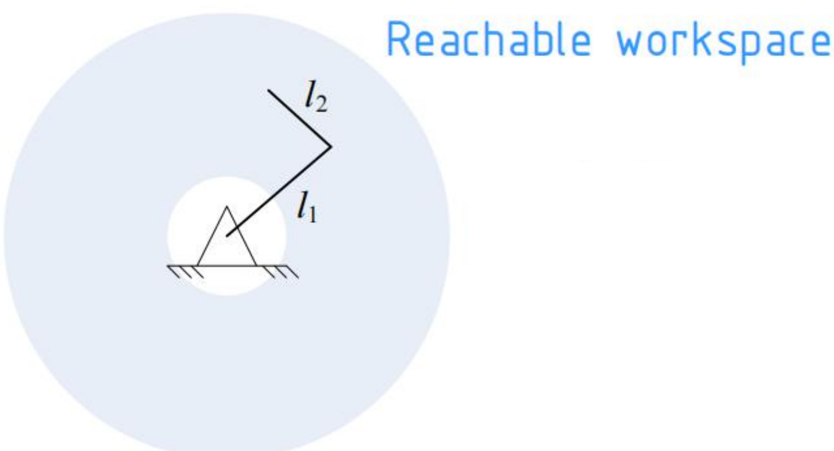
- Ex: A RR manipulator

  If $l_1 > l_2$



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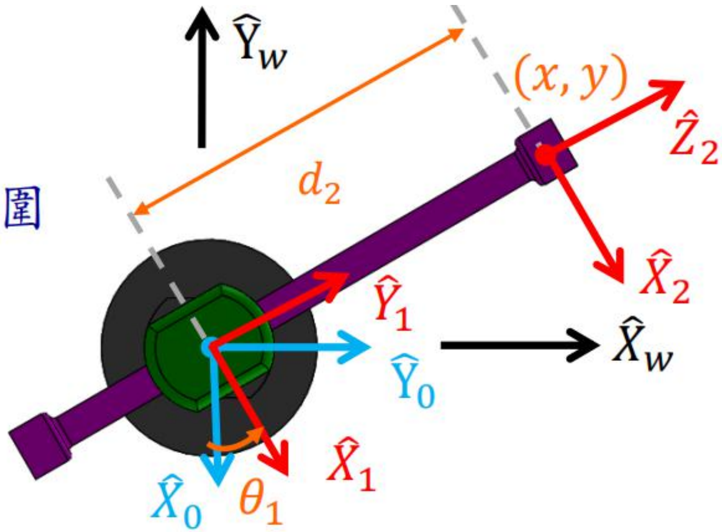
If $l_1 > l_2$ If $l_1 = l_2$

求解概念

- Subspace
  - 手臂在定義頭尾的T所能達到的變動範圍
- Ex: A RP manipulator
  - 2-DOF, Variables: (x, y)



$$
{}^{W}_{2}T = \begin{array}{c} \begin{array}{cccc} \phantom{\frac{y}{\sqrt{x^2+y^2}}} & & {}^{0}\hat{Z}_2 & {}^{0}P_{2\,ORG} \end{array} \\ \begin{bmatrix} \frac{y}{\sqrt{x^2+y^2}} & 0 & \frac{x}{\sqrt{x^2+y^2}} & x \\ \frac{-x}{\sqrt{x^2+y^2}} & 0 & \frac{y}{\sqrt{x^2+y^2}} & y \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}
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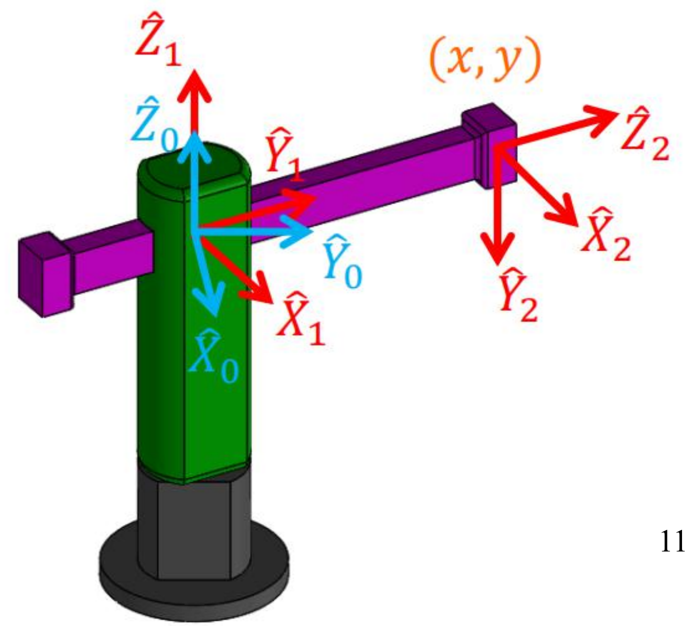

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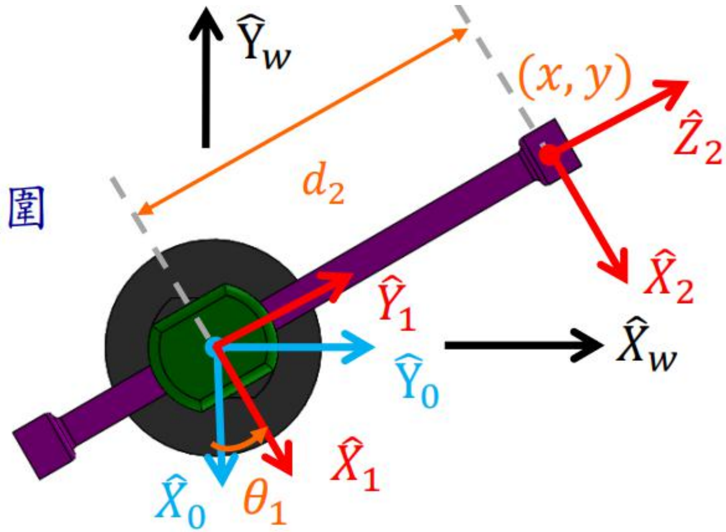

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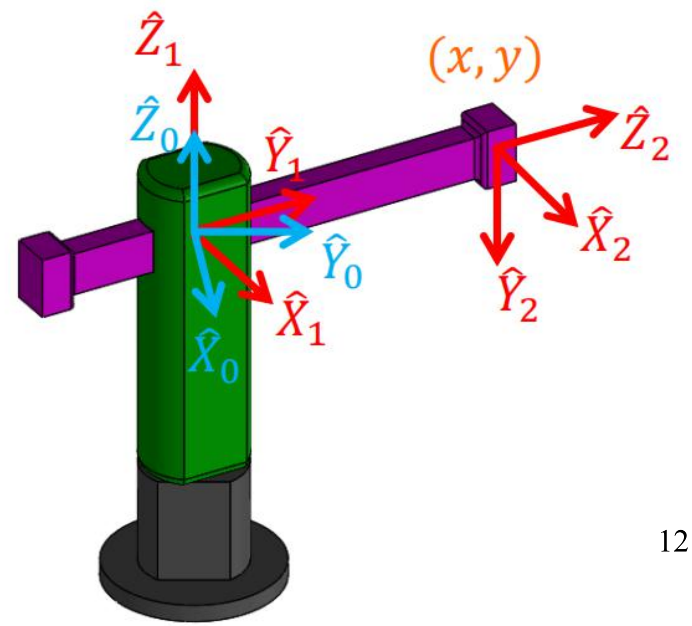

多重解

- 解的數目
  - 由於是nonlinear transcendental equations，6未知數6方程式不代表具有唯一解

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  - 由於是nonlinear transcendental equations，6未知數6方程式不代表具有唯一解
  - 是由joint數目和link參數所決定

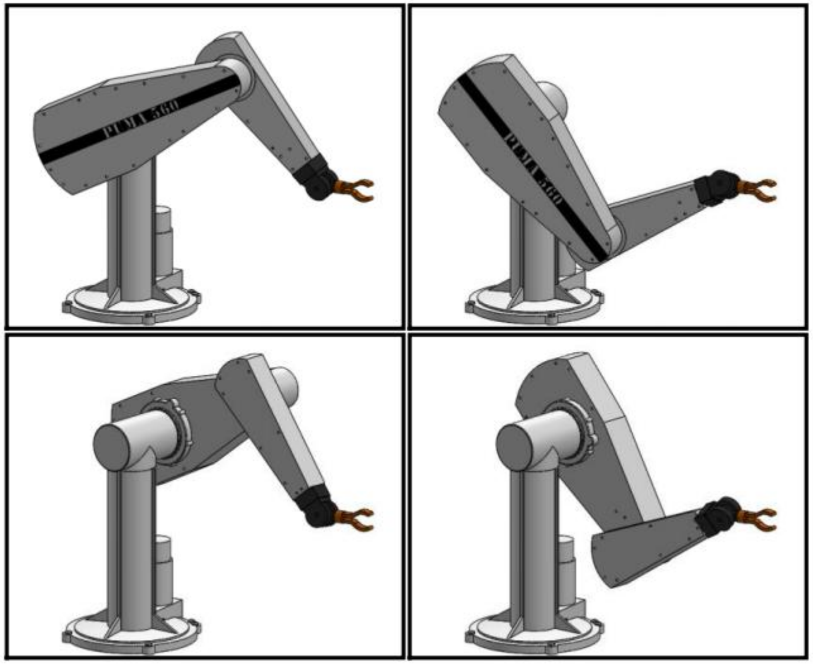
Ex: A RRRRRR manipulator

a_i	解的數目
$a_1 = a_3 = a_5 = 0$	≤ 4
$a_3 = a_5 = 0$	≤ 8
$a_3 = 0$	≤ 16
$All\ a_i \neq 0$	≤ 16

多重解

- Ex: PUMA (6 rotational joints)
  - 針對特定工作點，8組解
  - 前3軸具有4種姿態

  如右圖所示



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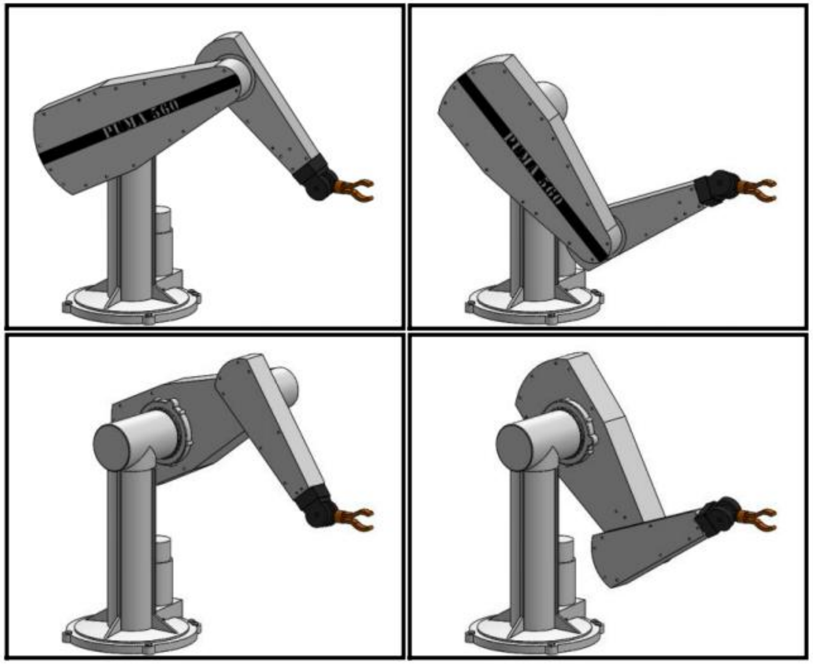
  - 每一個姿態中，具有2組手腕轉動姿態

$$\theta_4' = \theta_4 + 180^\circ$$

$$\theta_5' = -\theta_5$$

$$\theta_6' = \theta_6 + 180^\circ$$



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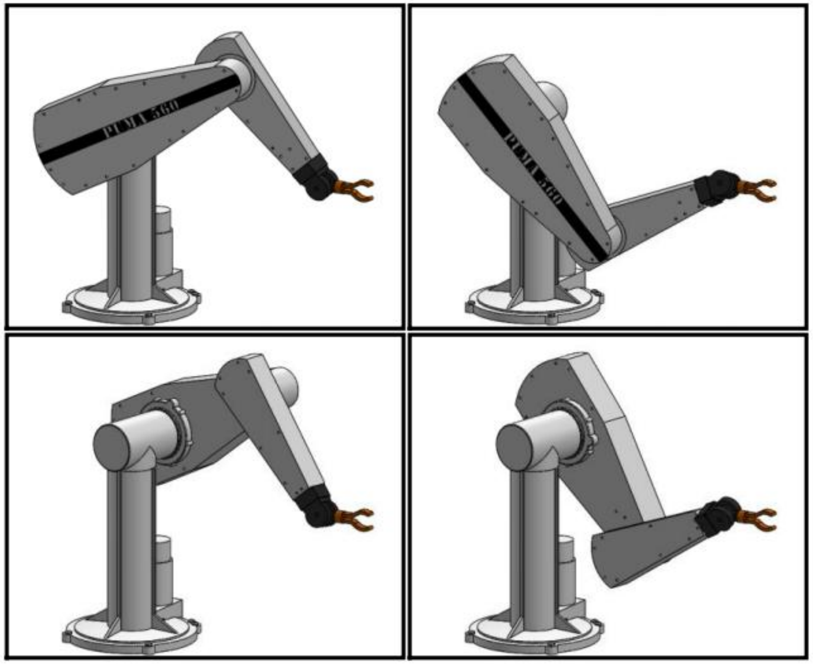
    $$\theta_4' = \theta_4 + 180^\circ$$

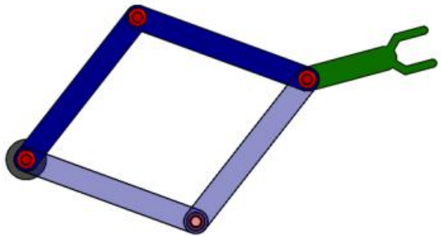
    $$\theta_5' = -\theta_5$$

    $$\theta_6' = \theta_6 + 180^\circ$$
  - 若手臂本身有幾何限制，並非每一種解都可以運作



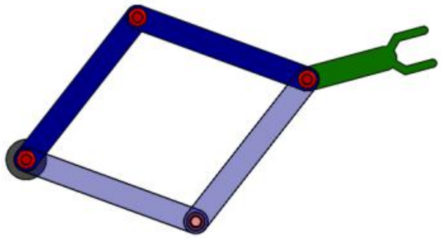
多重解

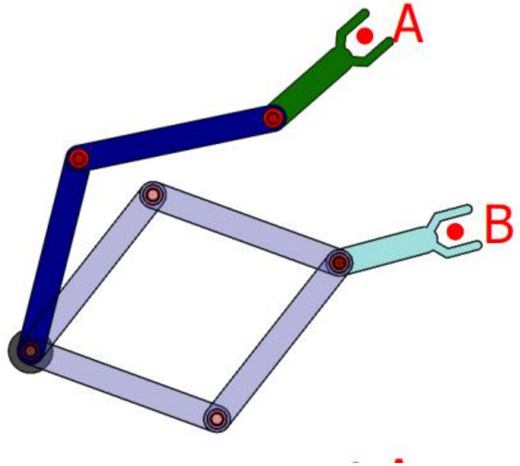
- 若具有多重解，解的選擇方式



多重解

- 若具有多重解，解的選擇方式
  - 離目前狀態最近的解
    - 最快
    - 最省能
    - ....





多重解

- 若具有多重解，解的選擇方式
  - 離目前狀態最近的解
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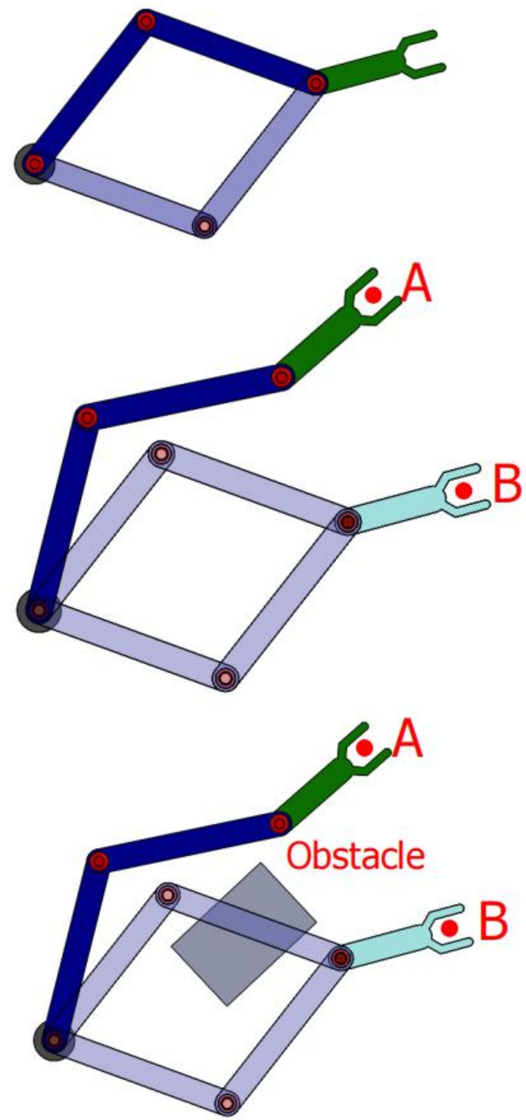
    - ....
  - 避開障礙物



求解方法

- 解析法 Closed-form solutions
  - 用 代數algebraic 或 幾何geometric 方法

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- 目前大多機械手臂設計成具有解析解
  - Pieper's solution: 相鄰三軸相交一點

例：RRR机械臂

- Ik problem: given (x, y, ϕ), $(\theta_1, \theta_2, \theta_3) = ?$
  - Forward kinematics

$$
{}^0_3T = \begin{bmatrix} c_{123} & -s_{123} & 0.0 & l_1 c_1 + l_2 c_{12} \\ s_{123} & c_{123} & 0.0 & l_1 s_1 + l_2 s_{12} \\ 0.0 & 0.0 & 1.0 & 0.0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
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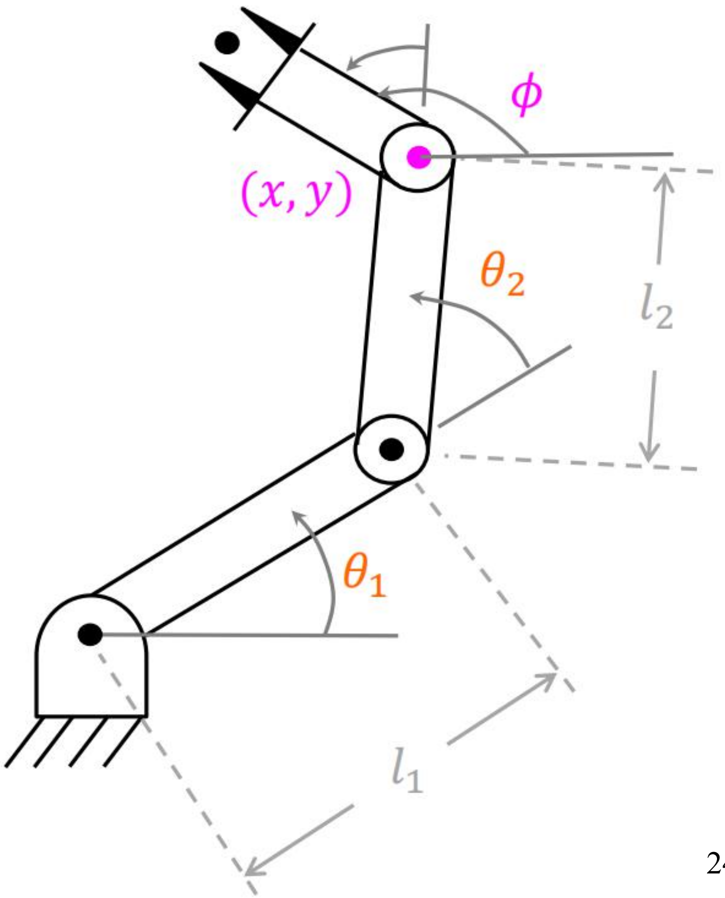

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$$

  - Goal point

$$
{}_3^0T = \begin{bmatrix} c_\phi & -s_\phi & 0.0 & x \\ s_\phi & c_\phi & 0.0 & y \\ 0.0 & 0.0 & 1.0 & 0.0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
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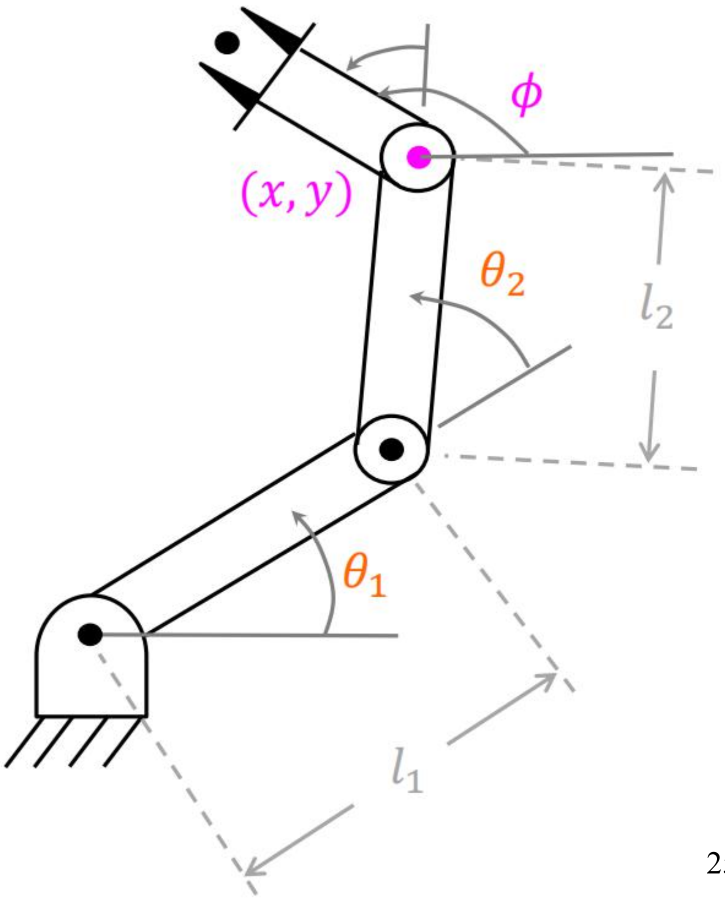
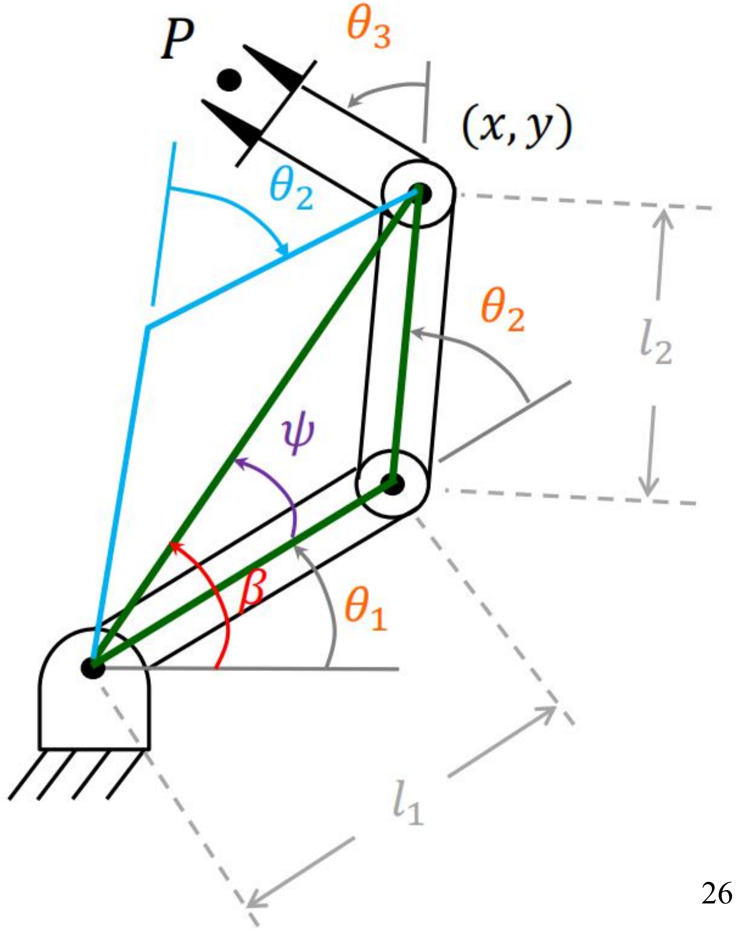


例：RRR机械臂

- 幾何法：將空間幾何切割成平面幾何

$$x^2+y^2=l_1^2+l_2^2-2l_1l_2\cos(180^\circ-\theta_2)$$

$$c_2=\frac{x^2+y^2-l_1^2-l_2^2}{2l_1l_2}$$



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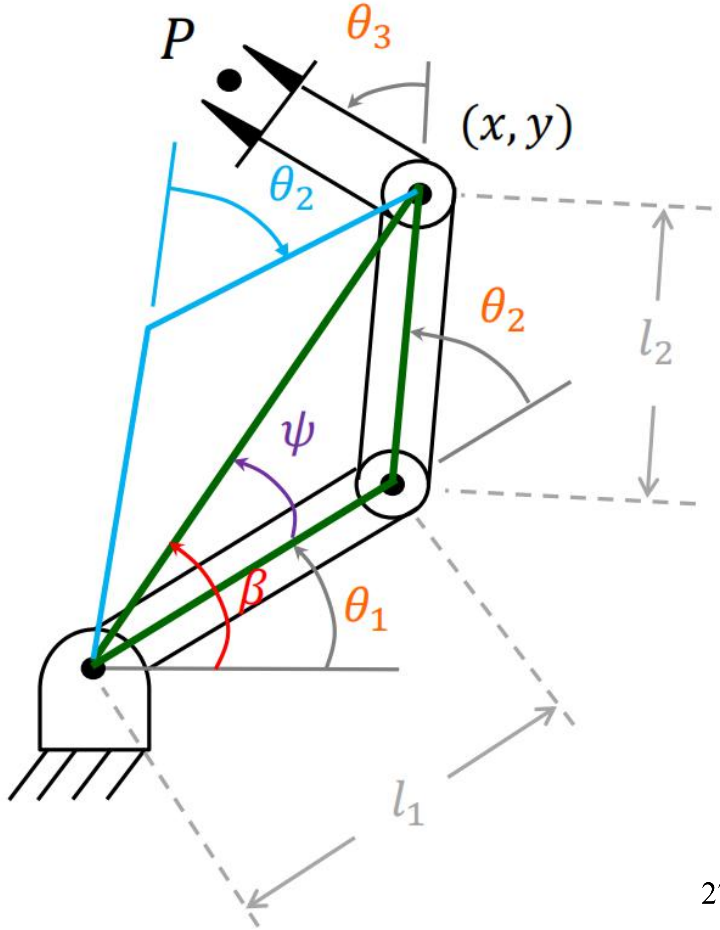
餘弦定理

$$cos\psi = \frac{l_2^2 - (x^2 + y^2) - l_1^2}{-2l_1\sqrt{x^2 + y^2}}$$

三角形內角 $0^\circ < \psi < 180^\circ$



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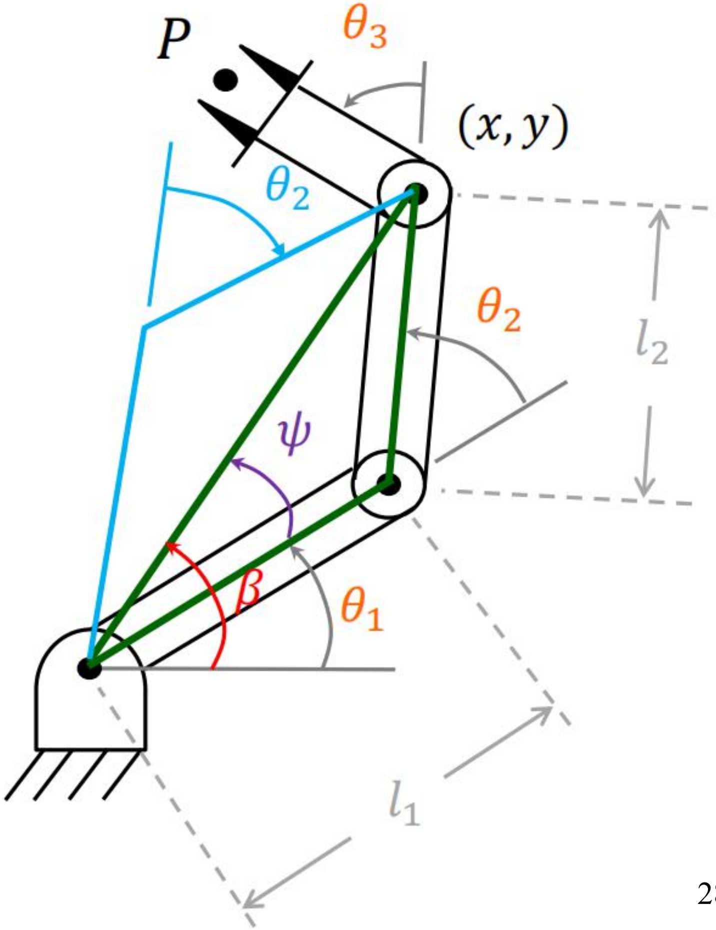
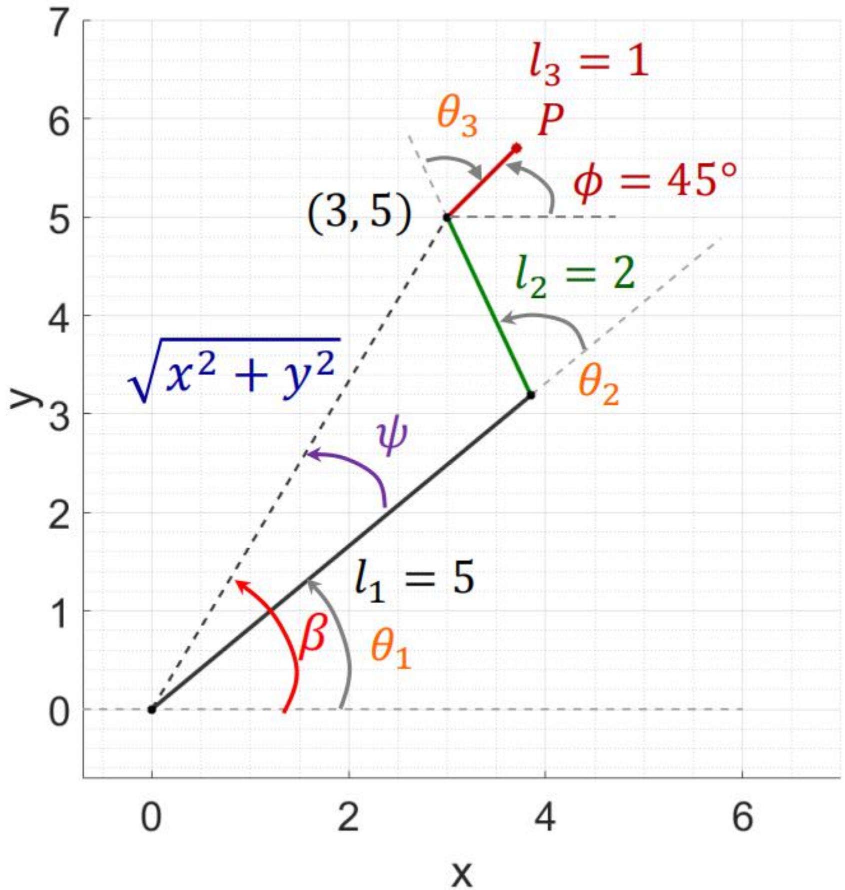
$$\theta_1=\begin{cases}atan2(y,x)+\psi & \theta_2<0^\circ\\ atan2(y,x)-\psi & \theta_2>0^\circ\end{cases}$$

$$\theta_3=\phi-\theta_1-\theta_2$$



例：RRR机械臂

- Ex: 量化計算



$$c_2 = \frac{x^2 + y^2 - l_1^2 - l_2^2}{2l_1 l_2}$$

$$\theta_2 = 75.5°$$

$$cos\psi = \frac{l_2^2 - (x^2 + y^2) - l_1^2}{-2l_1\sqrt{x^2 + y^2}}$$

$$\psi = 19.4°$$

$$\theta_1 = atan2(y, x) - \psi$$

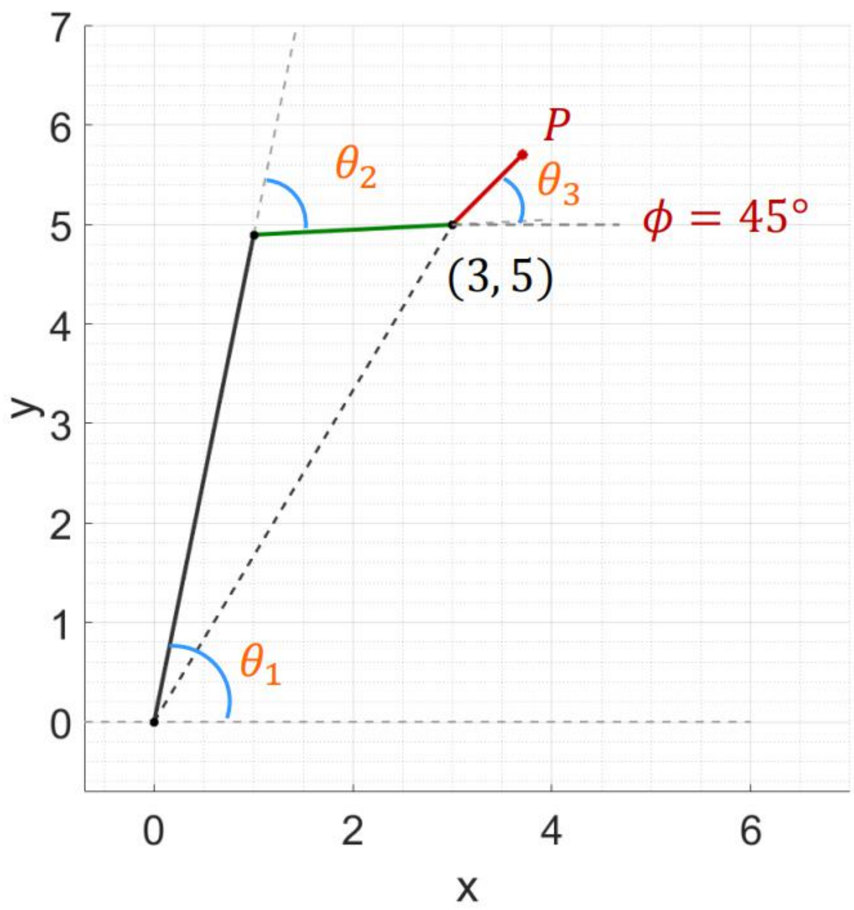
$$\theta_1 = 39.6°$$

$$\theta_3 = \phi - \theta_1 - \theta_2$$

$$\theta_3 = -70.2°$$

例：RRR机械臂

- In-Video Quiz: 針對同一個位移和姿態，求得另一組 $(\theta_1, \theta_2, \theta_3)$ 的解



(A)
$\theta_1 = 75.5$
$\theta_2 = -78.4$
$\theta_3 = 42.1$

(B)
$\theta_1 = 78.4$
$\theta_2 = -75.5$
$\theta_3 = 42.1$

(C)
$\theta_1 = -78.4$
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(D)
$\theta_1 = 59$
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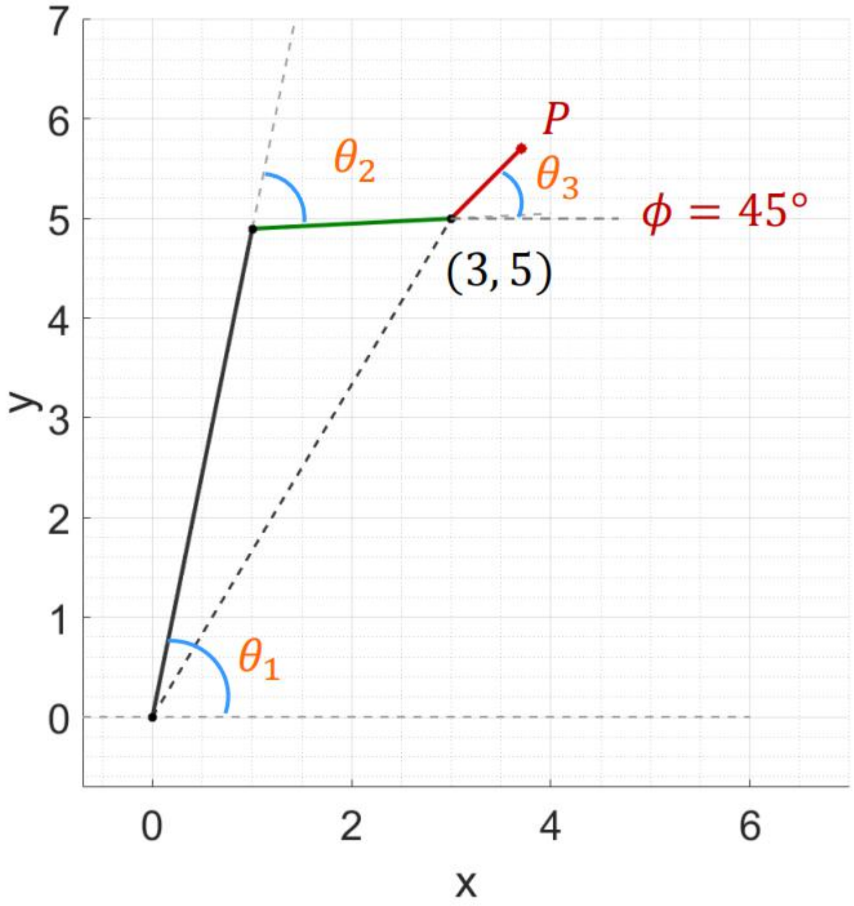

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例：RRR机械臂

- 代數解
  - 建立方程式

$$c_\phi = c_{123}$$

$$s_\phi = s_{123}$$

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$$

- 解 θ_2

$$x^2 + y^2 = {l_1}^2 + {l_2}^2 + 2 l_1 l_2 c_2$$

$$c_2 = \frac{x^2 + y^2 - {l_1}^2 - {l_2}^2}{2 l_1 l_2}$$

> 1 or < 1: too far for the manipulator to reach

$-1 \leq \quad \leq 1$: "two solutions" $\theta_2 = cos^{-1}(c_2)$

例：RRR机械臂

- 將求得的 θ_2 帶入方程式

$$x = l_1c_1 + l_2c_{12} = (l_1 + l_2c_2)c_1 + (-l_2s_2)s_1 \triangleq k_1c_1 - k_2s_1$$

$$y = l_1s_1 + l_2s_{12} = (l_1 + l_2c_2)s_1 + \quad (l_2s_2)c_1 \triangleq k_1s_1 + k_2c_1$$

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- 變數變換

define

$$r = +\sqrt{{k_1}^2 + {k_2}^2}$$

$$\gamma = Atan2(k_2, k_1)$$

then

$$k_1 = r\cos\gamma$$

$$k_2 = r\sin\gamma$$

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$$\gamma = Atan2(k_2, k_1)$$

then

$$k_1 = r\cos\gamma$$

$$k_2 = r\sin\gamma$$

And then

$$\frac{x}{r} = \cos\gamma\cos\theta_1 - \sin\gamma\sin\theta_1 = \cos(\gamma + \theta_1)$$

$$\frac{y}{r} = \cos\gamma\sin\theta_1 + \sin\gamma\cos\theta_1 = \sin(\gamma + \theta_1)$$

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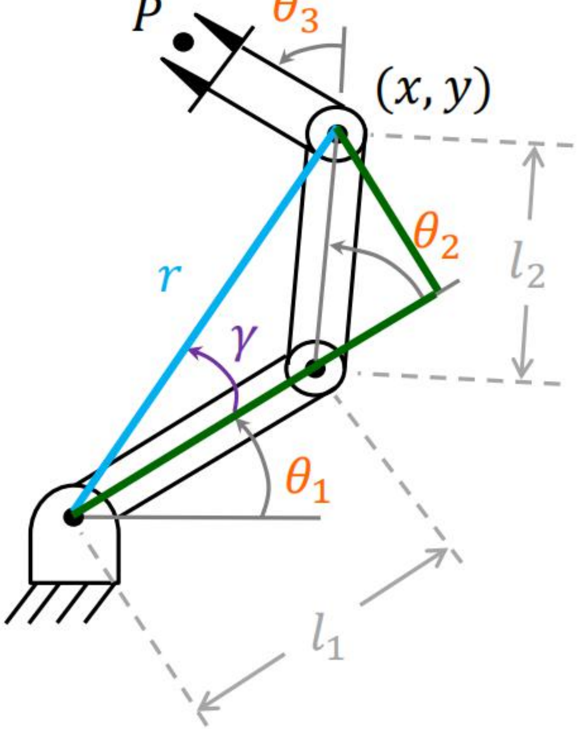
- 解 θ_1

$$\gamma + \theta_1 = Atan2\left(\frac{y}{r}, \frac{x}{r}\right) = Atan2(y, x)$$

$$\Rightarrow \theta_1 = Atan2(y, x) - Atan2(k_2, k_1)$$

當θ_2選不同解， $c2$和$s2$變動，k_1和k_2變動，θ_1也跟者變動



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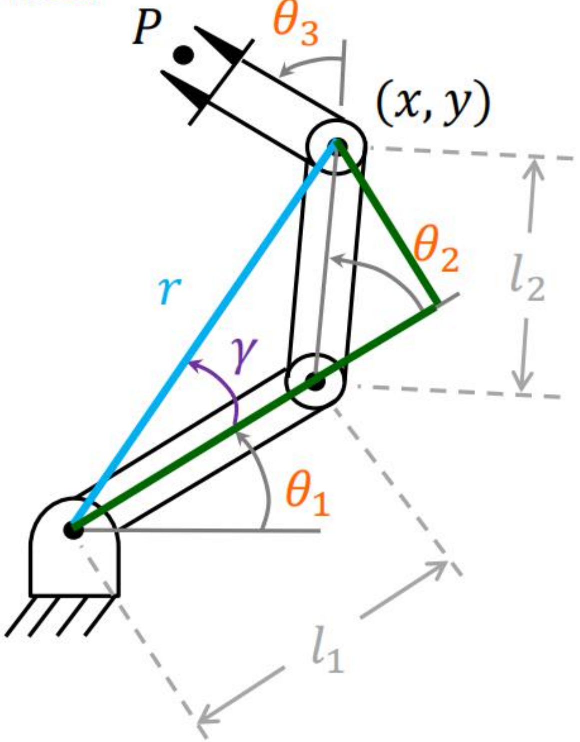

- 解 θ_3

$$\theta_1 + \theta_2 + \theta_3 = Atan2(s_\phi, c_\phi) = \phi$$

$$\Rightarrow \theta_3 = \phi - \theta_1 - \theta_2$$

三角函数方程式的求解

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$$acos\theta + \mathrm{b}\sin\theta = c$$

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$$(a+c)u^2 - 2bu + (c-a) = 0$$

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a, b, c大小有限制，不一定有解

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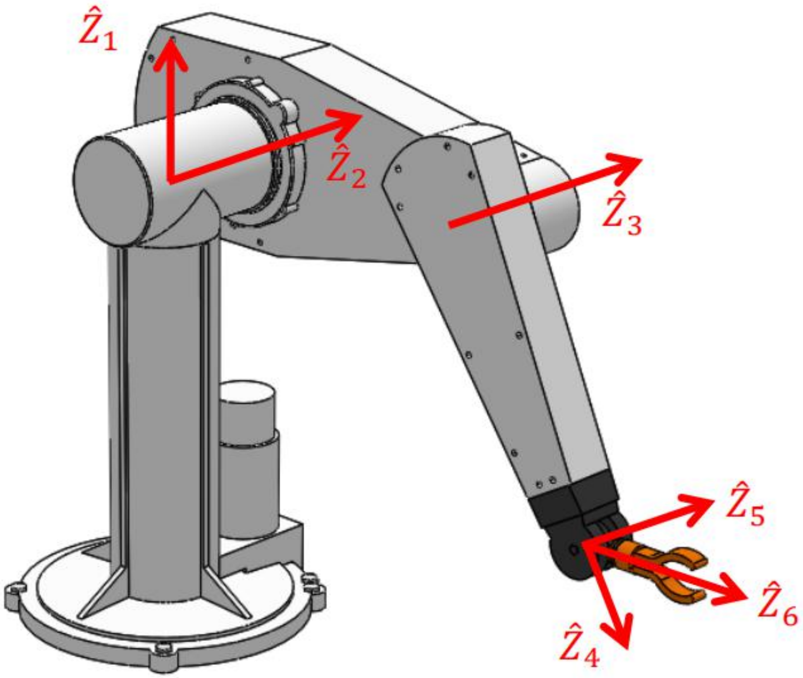
    $$\theta = 2\tan^{-1}\left(\frac{\mathrm{b} \pm \sqrt{b^2 + a^2 - c^2}}{\mathrm{a+c}}\right) \qquad a + c \neq 0$$

    $$\theta = 180^\circ \qquad a + c = 0$$

Pieper解

- 若6-DOF manipulator具有三個連續的軸交在同一點，則手臂有解析解



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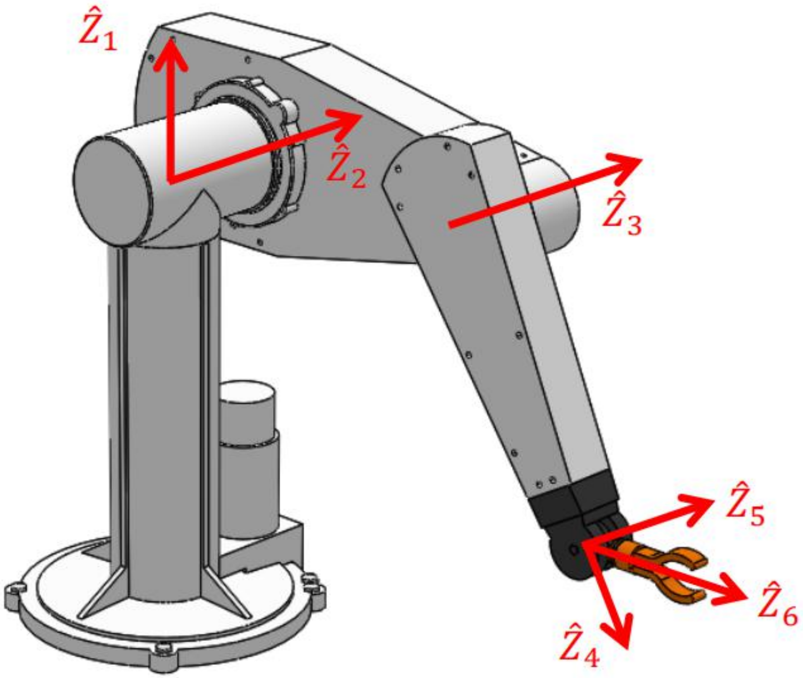

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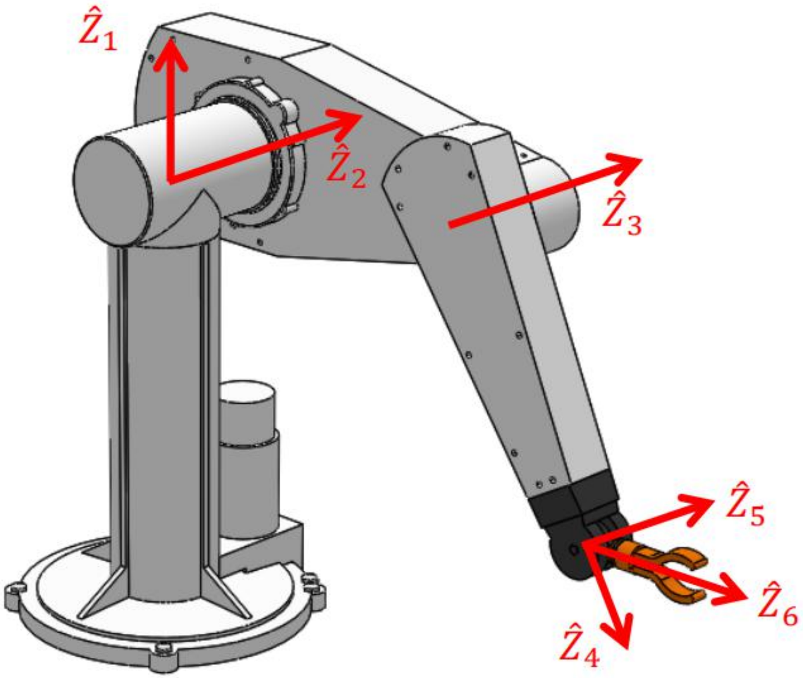
- Ex: A RRRRRR manipulator
  - 因後三軸交一點

$$^{0}P_{6\,ORG} = {}^{0}P_{4\,ORG}$$



Pieper解

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  - 法則：讓$\theta_1, \theta_2, \theta_3$層層分離

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Note: $$ {}^{i-1}_{i}T = \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ s\theta_i c\alpha_{i-1} & c\theta_i c\alpha_{i-1} & -s\alpha_{i-1} & -s\alpha_{i-1} d_i \\ s\theta_i s\alpha_{i-1} & c\theta_i s\alpha_{i-1} & c\alpha_{i-1} & c\alpha_{i-1} d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$\cos\theta_i = c\theta_i = c_i$$
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$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = {}^{0}P_{4\,ORG} = {}^{0}_{1}T\,{}^{1}_{2}T\,{}^{2}_{3}T\;{}^{3}P_{4\,ORG}$$

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$\cos\theta_i = c\theta_i = c_i$
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$$= {}^{0}_{1}T\,{}^{1}_{2}T\,{}^{2}_{3}T \begin{bmatrix} a_3 \\ -d_4 s\alpha_3 \\ d_4 c\alpha_3 \\ 1 \end{bmatrix} = {}^{0}_{1}T\,{}^{1}_{2}T \begin{bmatrix} f_1(\theta_3) \\ f_2(\theta_3) \\ f_3(\theta_3) \\ 1 \end{bmatrix}$$

4^{th} column of ${}^{3}_{4}T$

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so

4^{th} column of ${}^{3}_{4}T$

$$\begin{bmatrix} f_1(\theta_3) \\ f_2(\theta_3) \\ f_3(\theta_3) \\ 1 \end{bmatrix} = {}^{2}_{3}T \begin{bmatrix} a_3 \\ -d_4 s\alpha_3 \\ d_4 c\alpha_3 \\ 1 \end{bmatrix}$$

讓$\theta_1, \theta_2, \theta_3$層層分離，$f$為$\theta_3$函數

$$f_1(\theta_3) = a_3 c_3 + d_4 s\alpha_3 s_3 + a_2$$
$$f_2(\theta_3) = a_3 c\alpha_2 s_3 - d_4 s\alpha_3 c\alpha_2 c_3 - d_4 s\alpha_2 c\alpha_3 - d_3 s\alpha_2$$
$$f_3(\theta_3) = a_3 s\alpha_2 s_3 - d_4 s\alpha_3 s\alpha_2 c_3 + d_4 c\alpha_2 c\alpha_3 + d_3 c\alpha_2$$

Pieper解

- 下一步

$$ {}^{0}P_{4ORG} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = {}^{0}_{1}T\, {}^{1}_{2}T \begin{bmatrix} f_1(\theta_3) \\ f_2(\theta_3) \\ f_3(\theta_3) \\ 1 \end{bmatrix} = {}^{0}_{1}T \begin{bmatrix} g_1(\theta_2,\theta_3) \\ g_2(\theta_2,\theta_3) \\ g_3(\theta_2,\theta_3) \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 g_1 - s_1 g_2 \\ s_1 g_1 + c_1 g_2 \\ g_3 \\ 1 \end{bmatrix} $$

讓$\theta_1, \theta_2, \theta_3$層層分離， g為θ_2, θ_3函數

$$ g_1(\theta_2,\theta_3) = c_2 f_1 - s_2 f_2 + a_1 $$
$$ g_2(\theta_2,\theta_3) = s_2 c\alpha_1 f_1 + c_2 c\alpha_1 f_2 - s\alpha_1 f_3 - d_2 s\alpha_1 $$
$$ g_3(\theta_2,\theta_3) = s_2 s\alpha_1 f_1 + c_2 s\alpha_1 f_2 + c\alpha_1 f_3 + d_2 c\alpha_1 $$

Pieper解

- 下一步

$$
{}^{0}P_{4ORG} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = {}^{0}_{1}T\,{}^{1}_{2}T \begin{bmatrix} f_1(\theta_3) \\ f_2(\theta_3) \\ f_3(\theta_3) \\ 1 \end{bmatrix} = {}^{0}_{1}T \begin{bmatrix} g_1(\theta_2,\theta_3) \\ g_2(\theta_2,\theta_3) \\ g_3(\theta_2,\theta_3) \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 g_1 - s_1 g_2 \\ s_1 g_1 + c_1 g_2 \\ g_3 \\ 1 \end{bmatrix}
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$$
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\end{aligned}
$$

$$
\begin{aligned}
r &= x^2 + y^2 + z^2 = g_1^2 + g_2^2 + g_3^2 \\
&= f_1^2 + f_2^2 + f_3^2 + a_1^2 + d_2^2 + 2d_2 f_3 + 2a_1(c_2 f_1 - s_2 f_2) \\
&= (k_1 c_2 + k_2 s_2) 2a_1 + k_3
\end{aligned}
$$

r僅為θ_2, θ_3函數

$$
\begin{aligned}
k_1(\theta_3) &= f_1 \\
k_2(\theta_3) &= -f_2 \\
k_3(\theta_3) &= f_1^2 + f_2^2 + f_3^2 + a_1^2 + d_2^2 + 2d_2 f_3
\end{aligned}
$$

Pieper解

- 此外

$$z = g_3 = (k_1 s_2 - k_2 c_2) s\alpha_1 + k_4$$

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- 整合 r 和 z 一起考量

$$\begin{cases} r = (k_1 c_2 + k_2 s_2) 2a_1 + k_3 \\ z = (k_1 s_2 - k_2 c_2) s\alpha_1 + k_4 \end{cases}$$

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  - Else

$$\frac{(r - k_3)^2}{4a_1^2} + \frac{(z - k_4)^2}{s^2\alpha_1} = k_1^2 + k_2^2$$

Pieper解

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⇨ Solve θ_3 of all three cases by using "$u = \tan\left(\frac{\theta_3}{2}\right)$"

Pieper解

- 最後

Using $r = (k_1 c_2 + k_2 s_2) 2a_1 + k_3$ to solve θ_2

Using $x = c_1 g_1(\theta_2, \theta_3) - s_1 g_2(\theta_2, \theta_3)$ to solve θ_1

Pieper解

- 最後

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  Using $x = c_1g_1(\theta_2, \theta_3) - s_1g_2(\theta_2, \theta_3)$ to solve θ_1

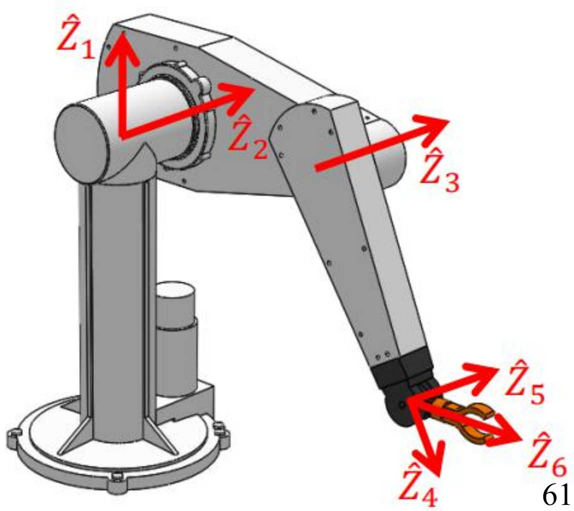
- Orientation
  - θ_1, θ_2, θ_3 已知

    $${}^{3}_{6}R = {}^{0}_{3}R^{-1}\,{}^{0}_{6}R$$

  - 以 Z-Y-Z Euler angle 求解 θ_4, θ_5, θ_6

Pieper解

- Joints 4-6, DH definition



Pieper解

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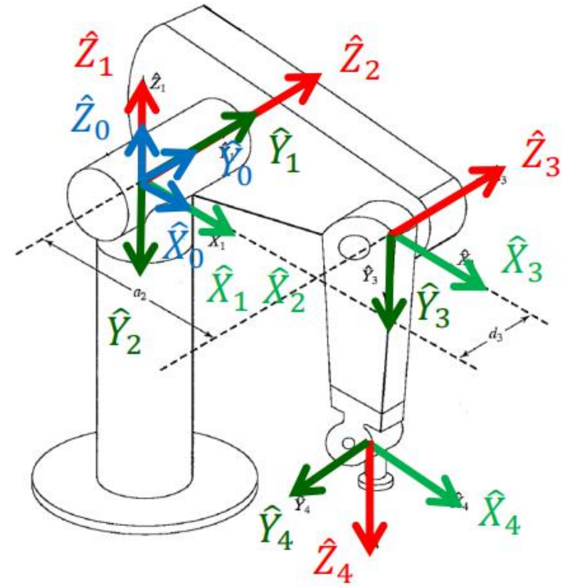

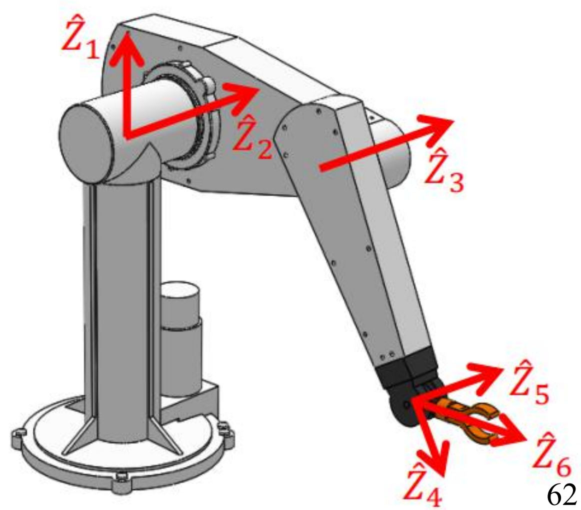

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Pieper解

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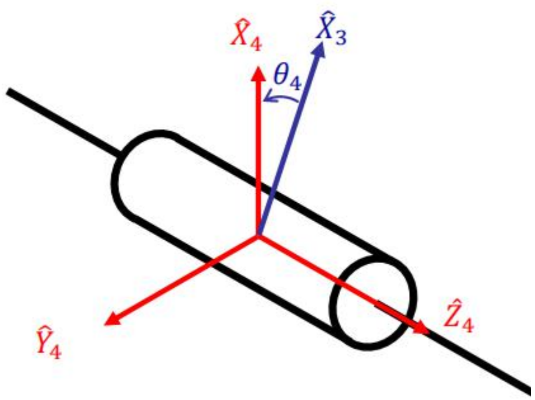
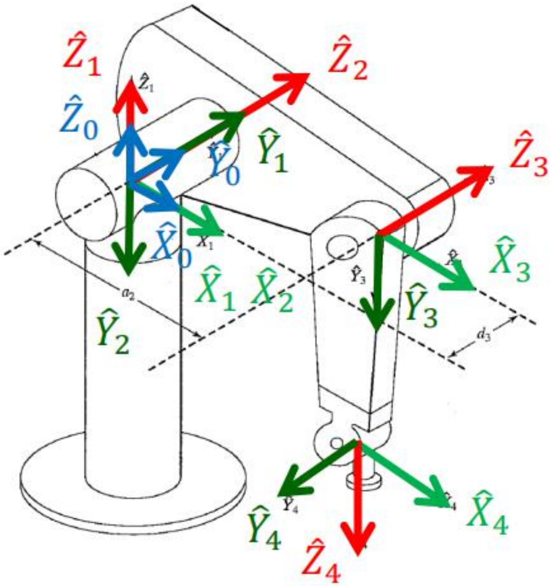
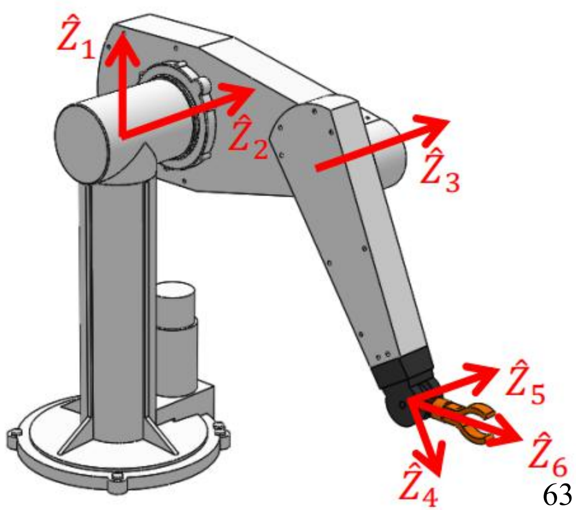
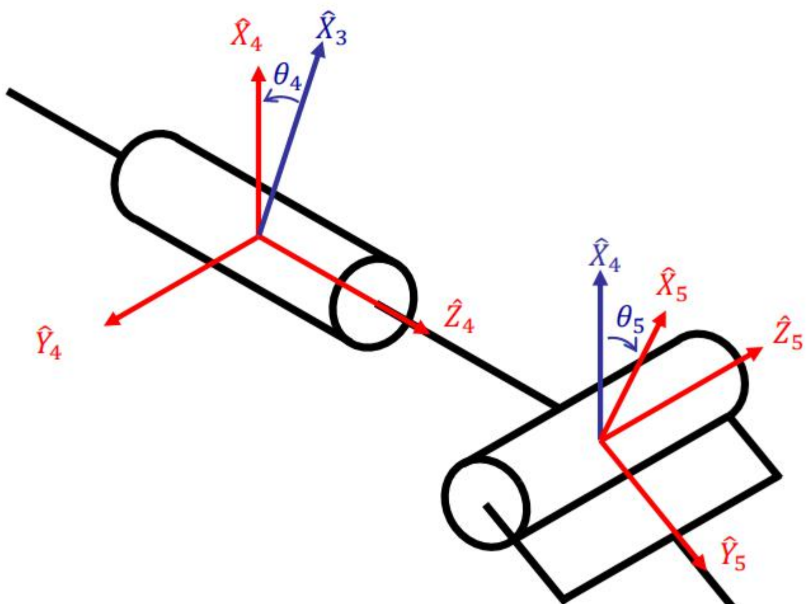


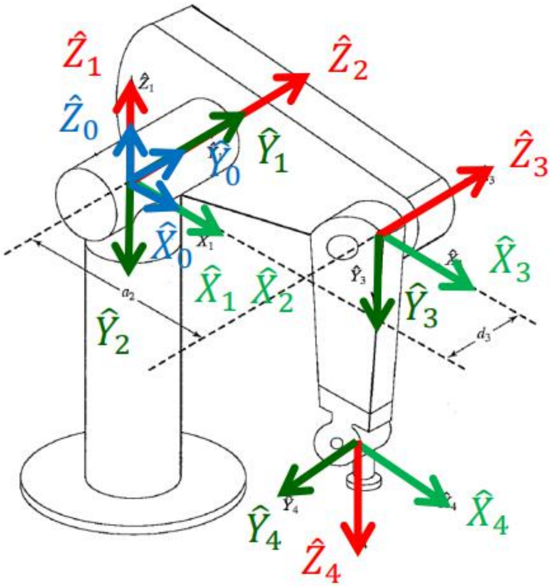


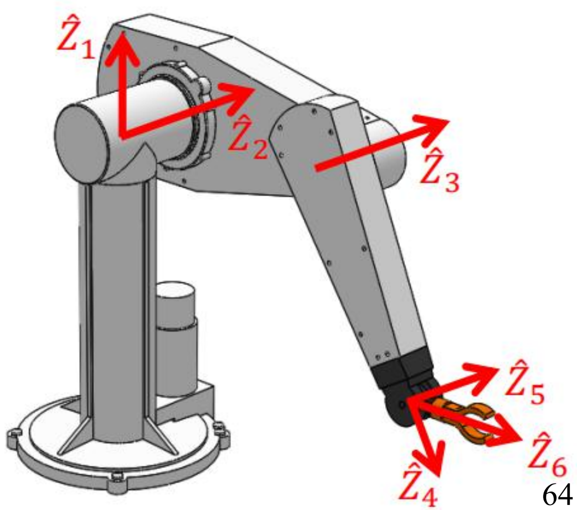


Pieper解

- Joints 4-6, DH definition







Pieper解

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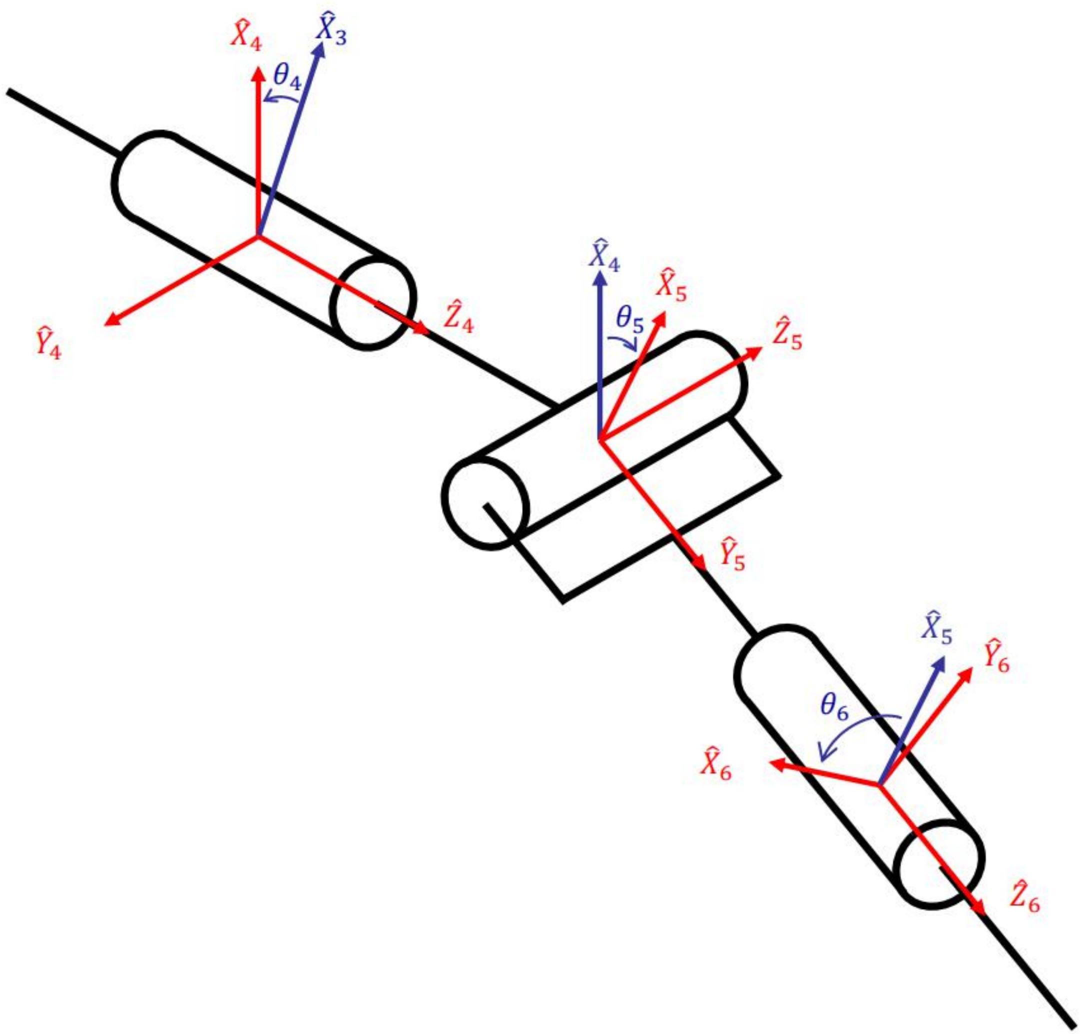
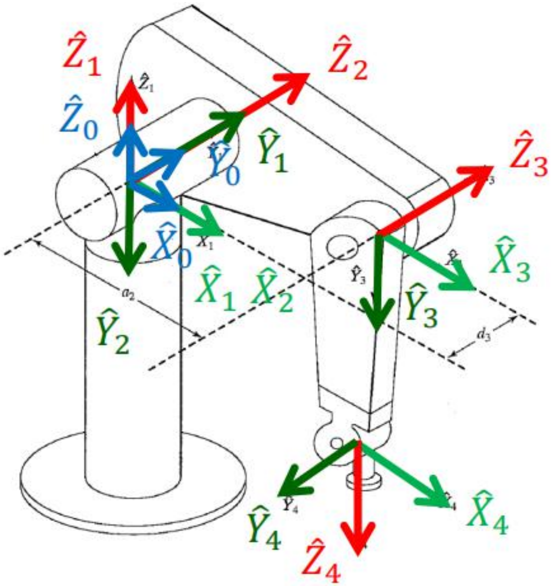
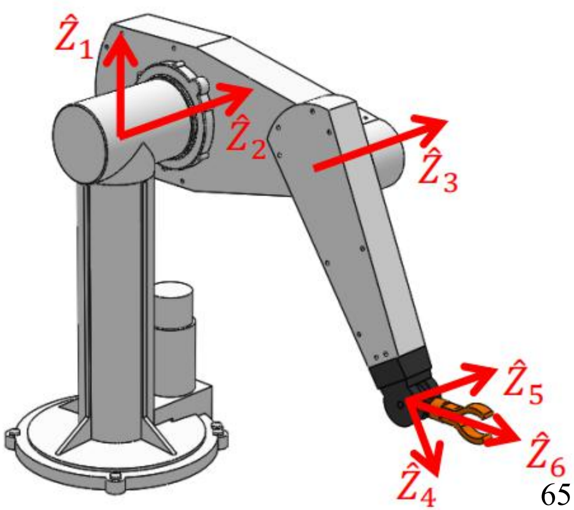
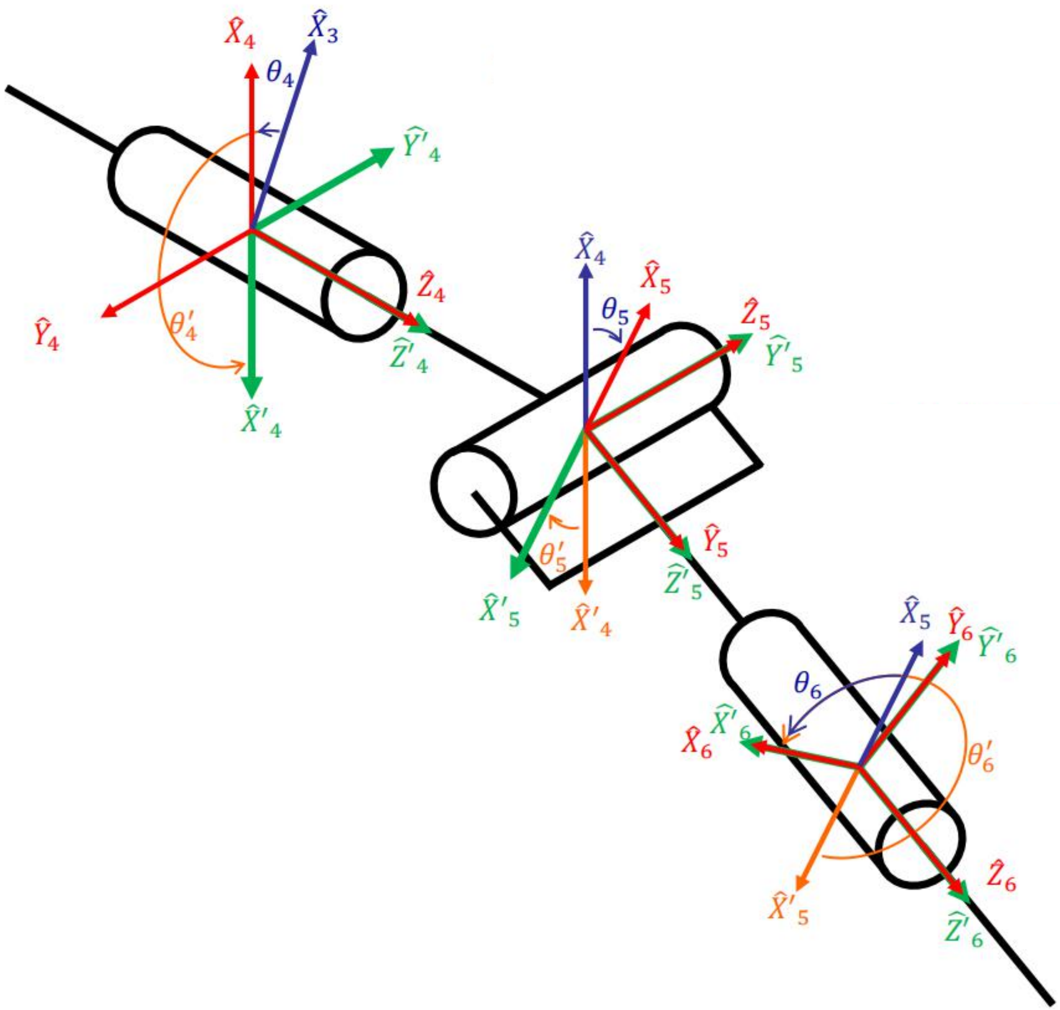






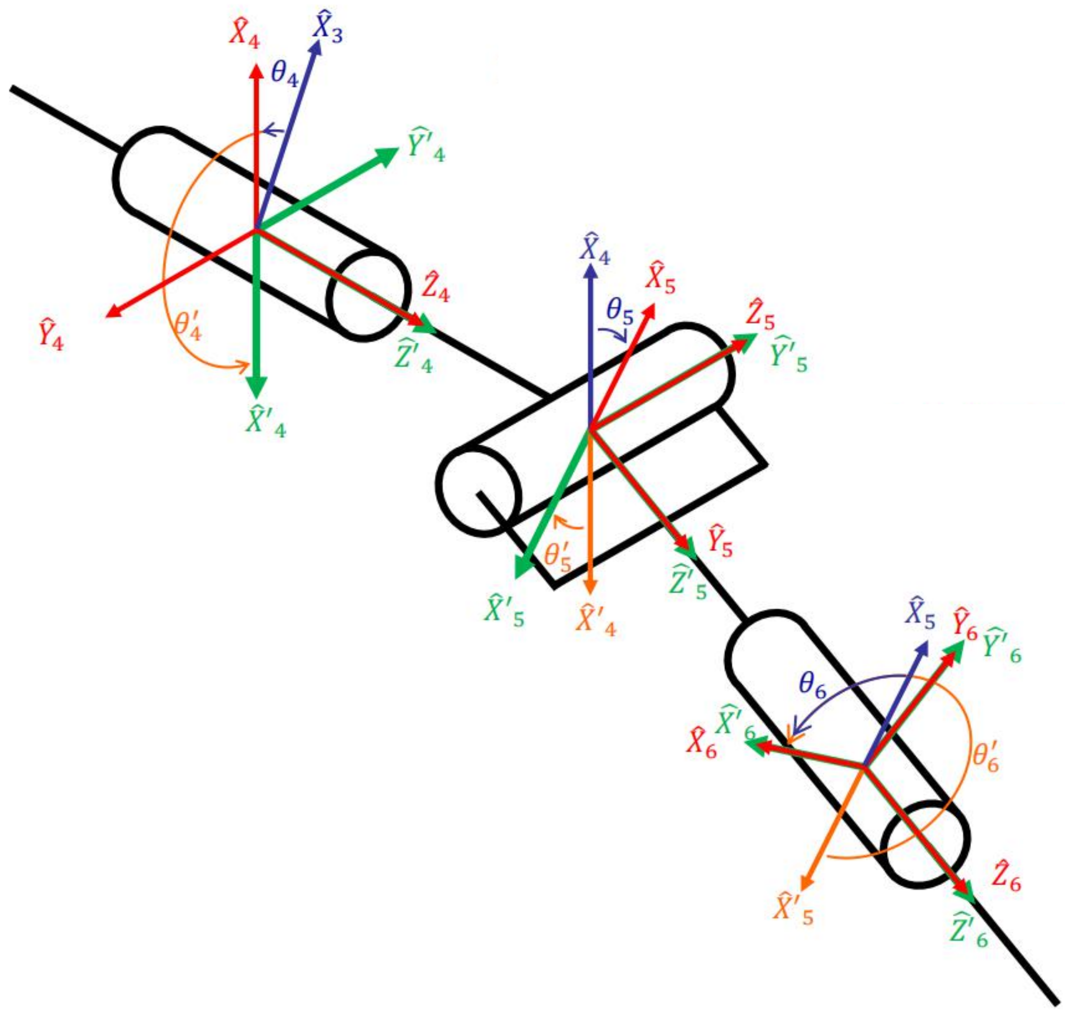
Pieper解

- DH definition vs. Z-Y-Z Euler Angles



Pieper解

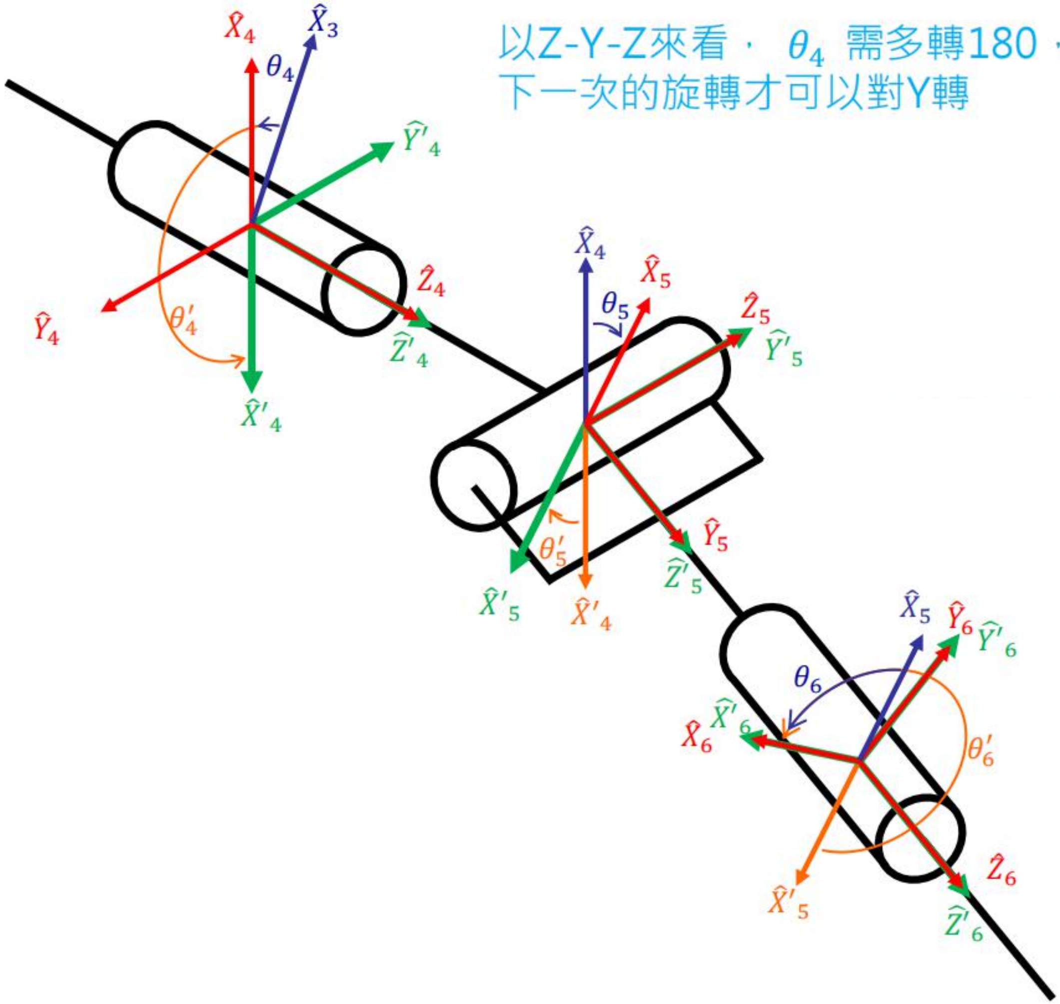
❑ DH definition vs. Z-Y-Z Euler Angles

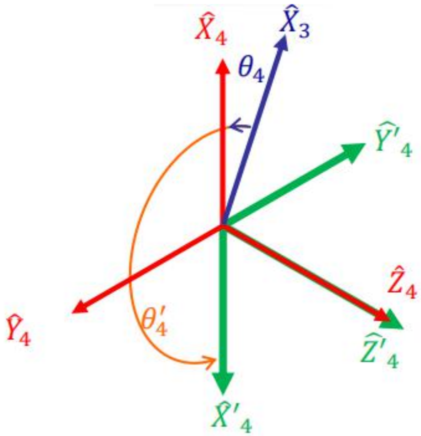


Pieper解

- DH definition vs. Z-Y-Z Euler Angles

以Z-Y-Z來看， θ_4 需多轉180，
下一次的旋轉才可以對Y轉





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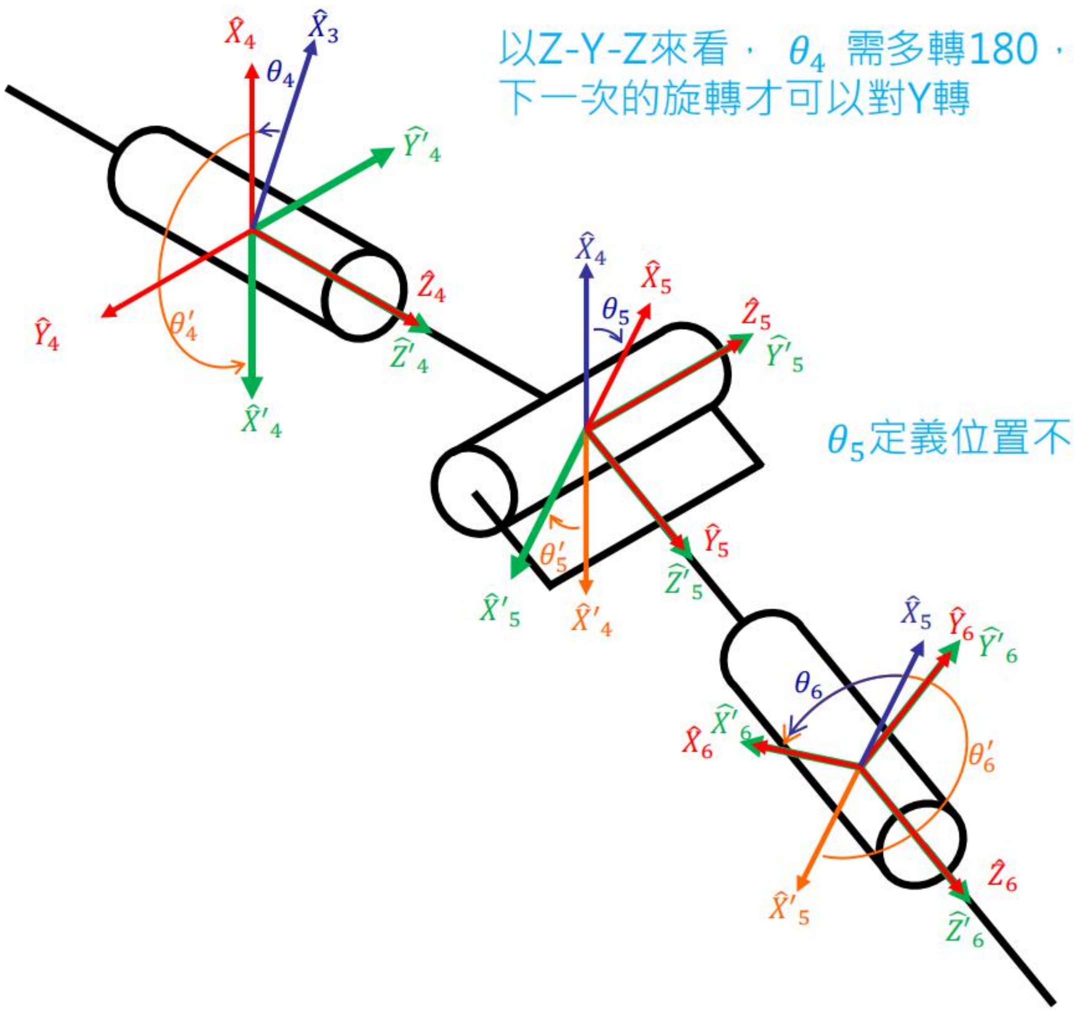

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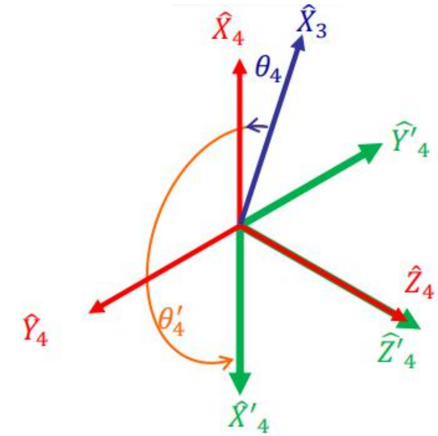
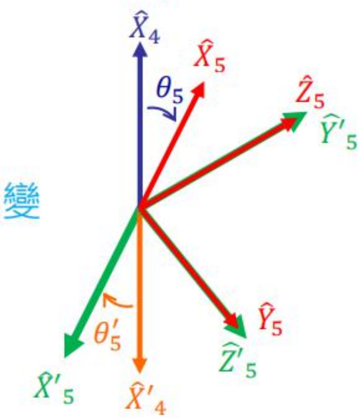


θ_5定義位置不同，但量值沒變



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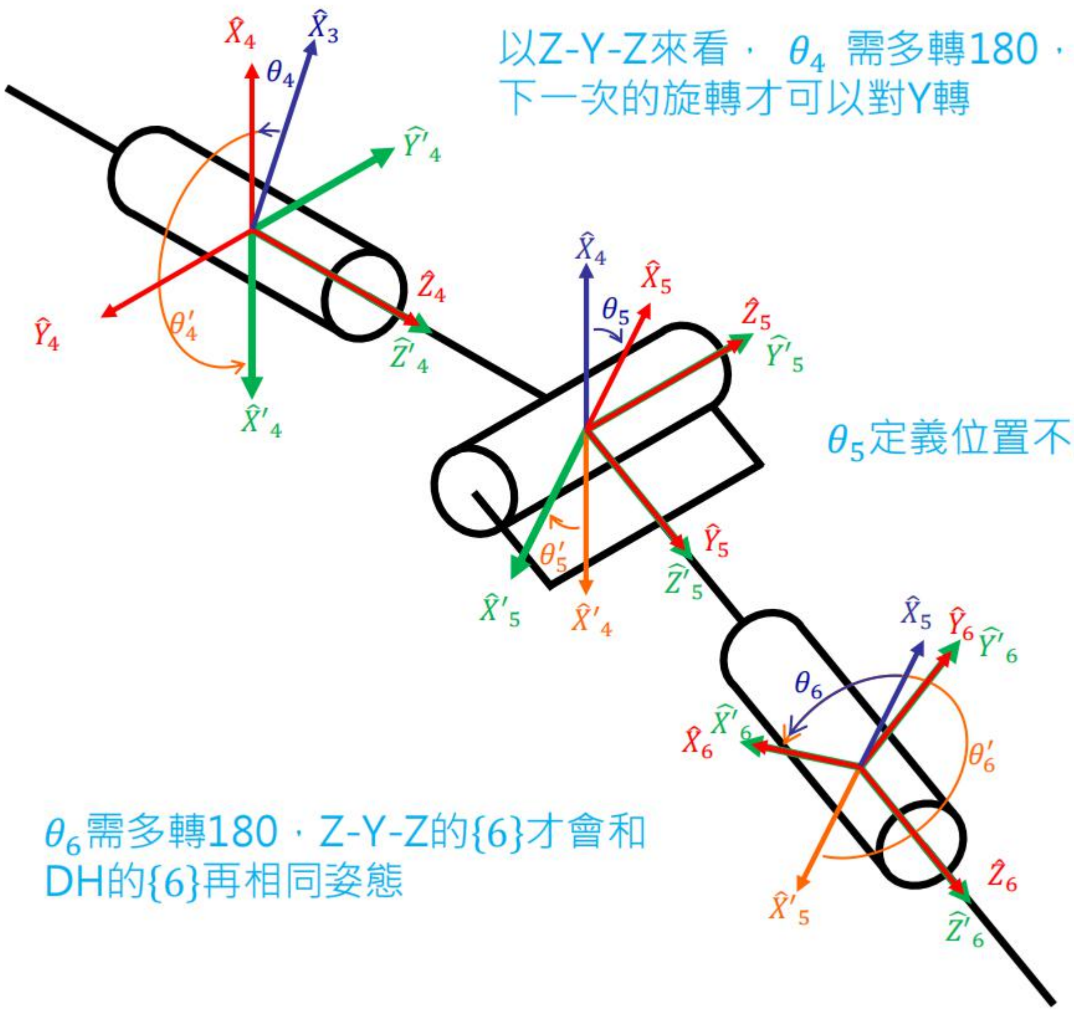

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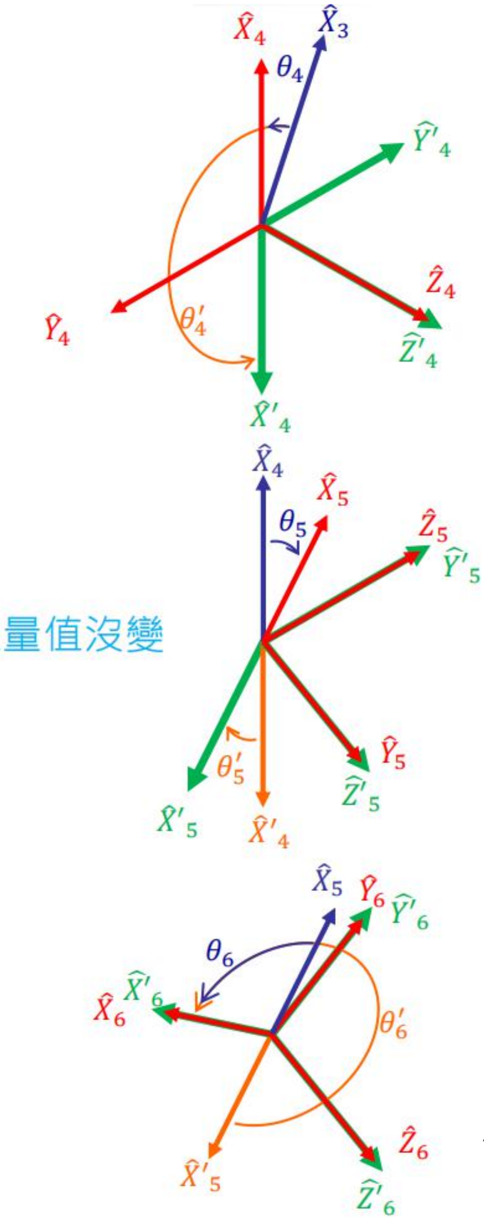
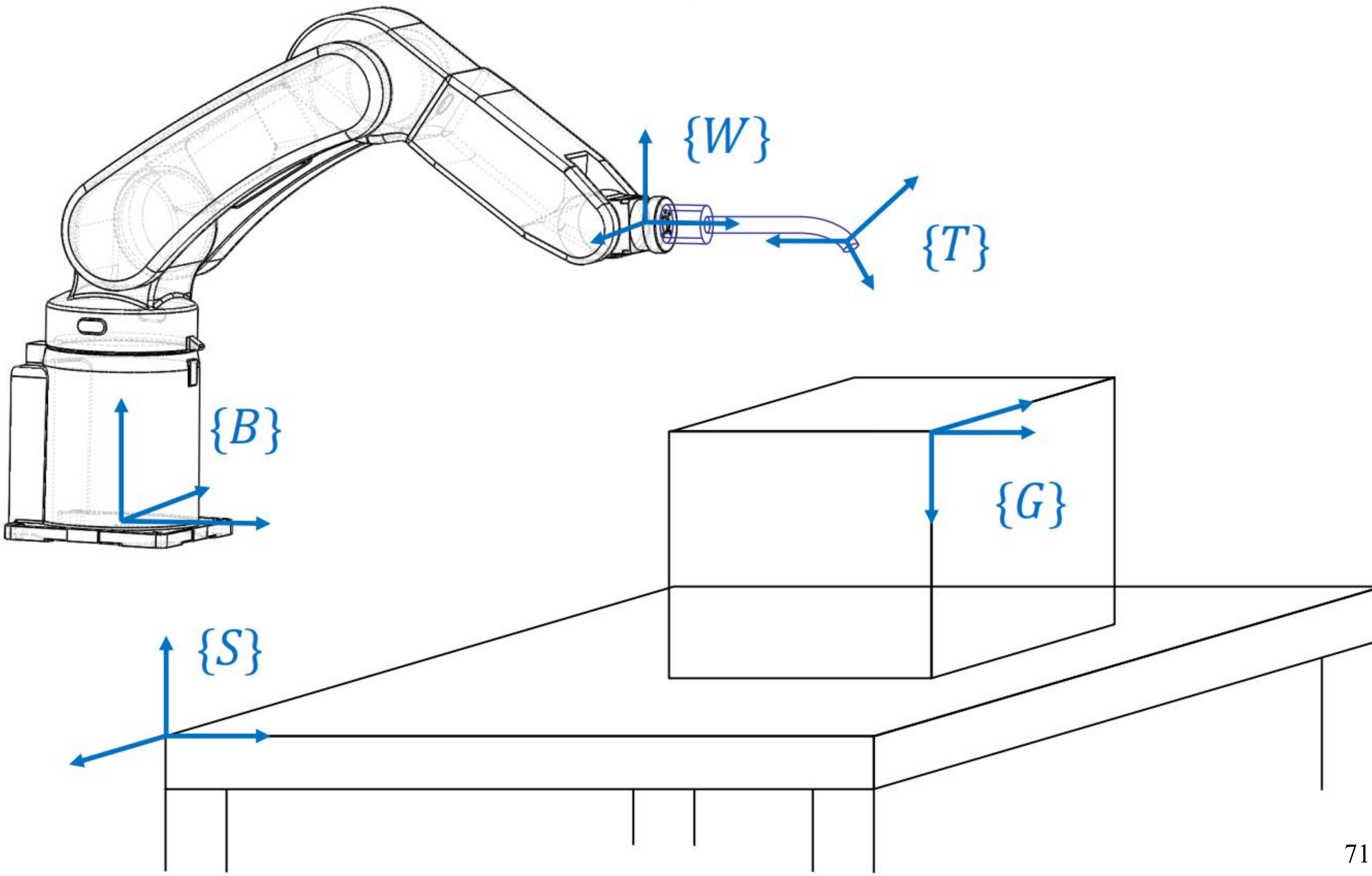


θ_6需多轉180，Z-Y-Z的{6}才會和
DH的{6}再相同姿態

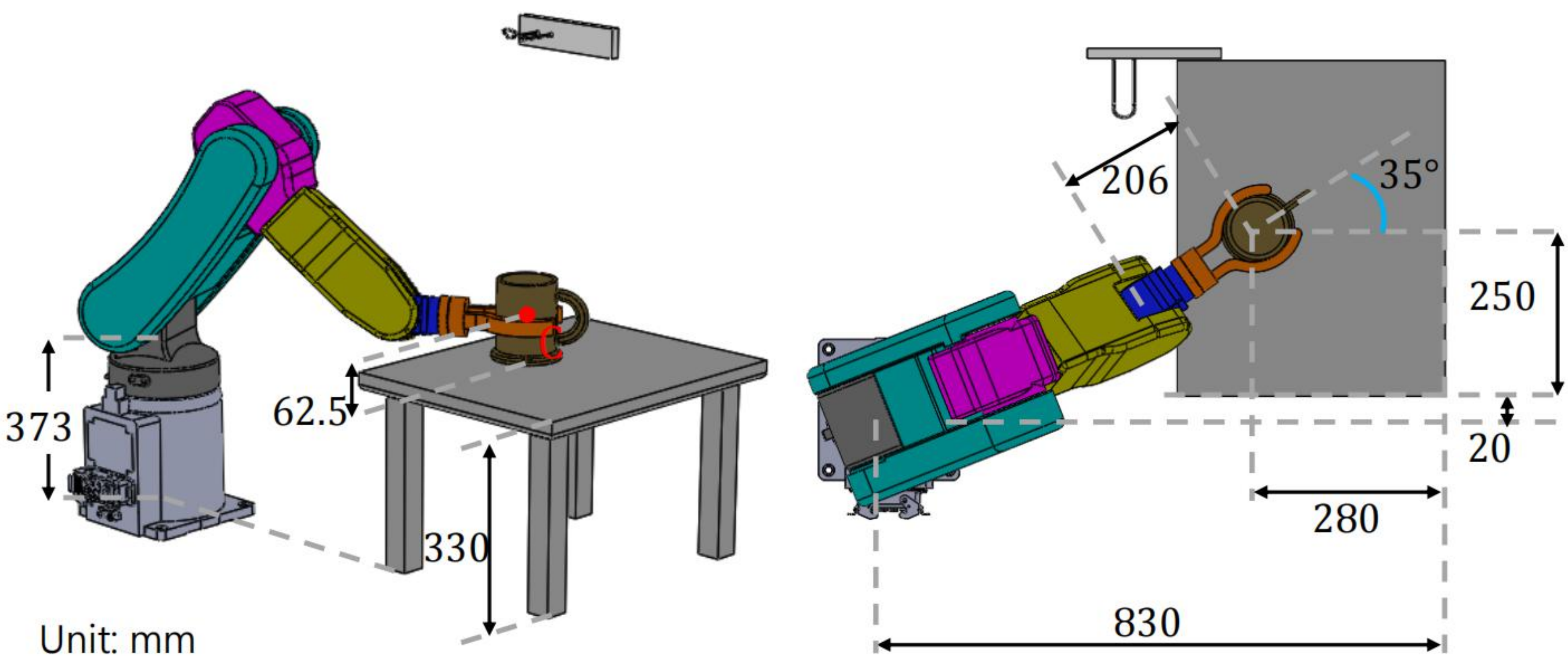
坐标系

- Base, wrist, tool, station, and goal frames



例：物件抓取任务

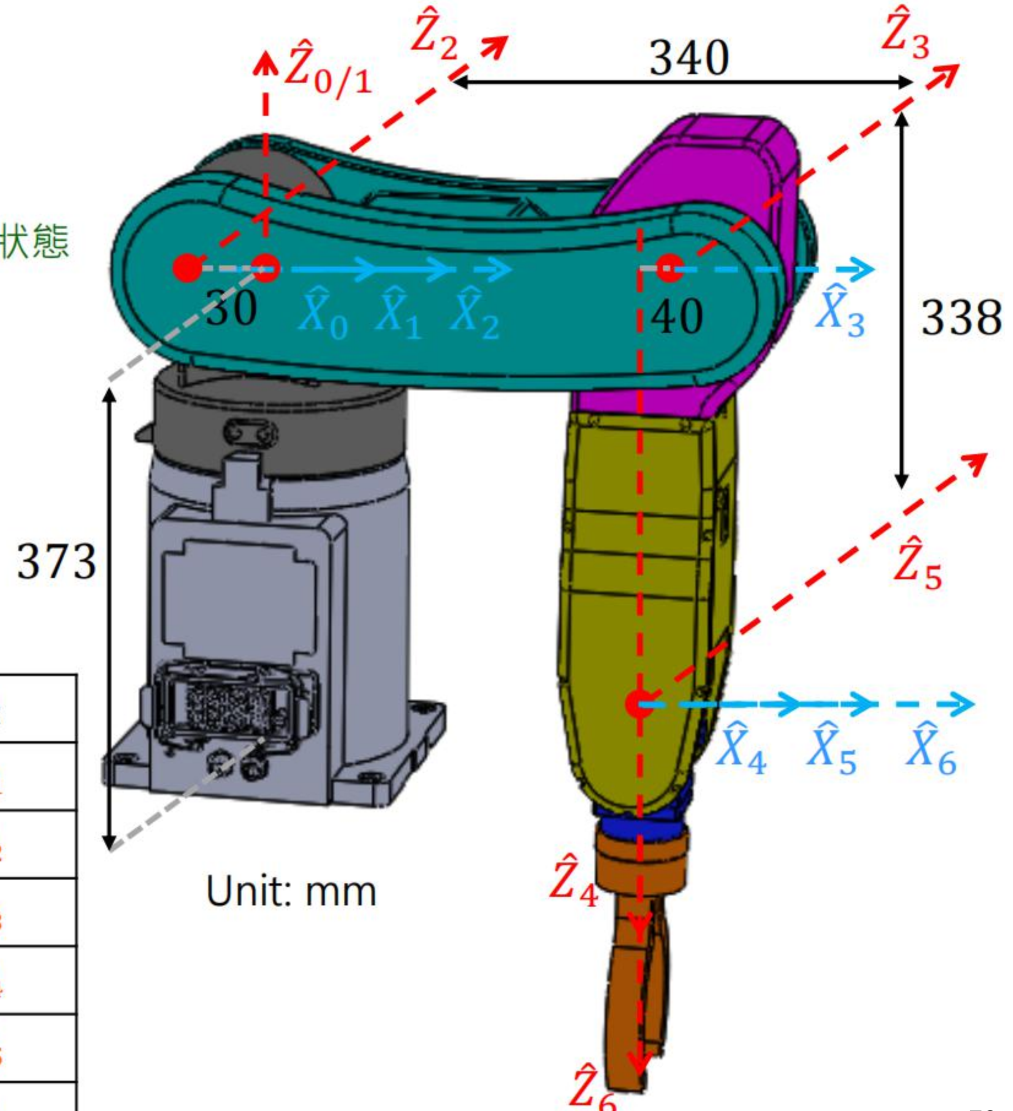
- 現階段任務：為使RRRRRR手臂能以下圖姿態夾住杯子（任務的起始點C），手臂的6個joint angles需為何?



例：物件抓取任务

- Step 1: 定義DH Table

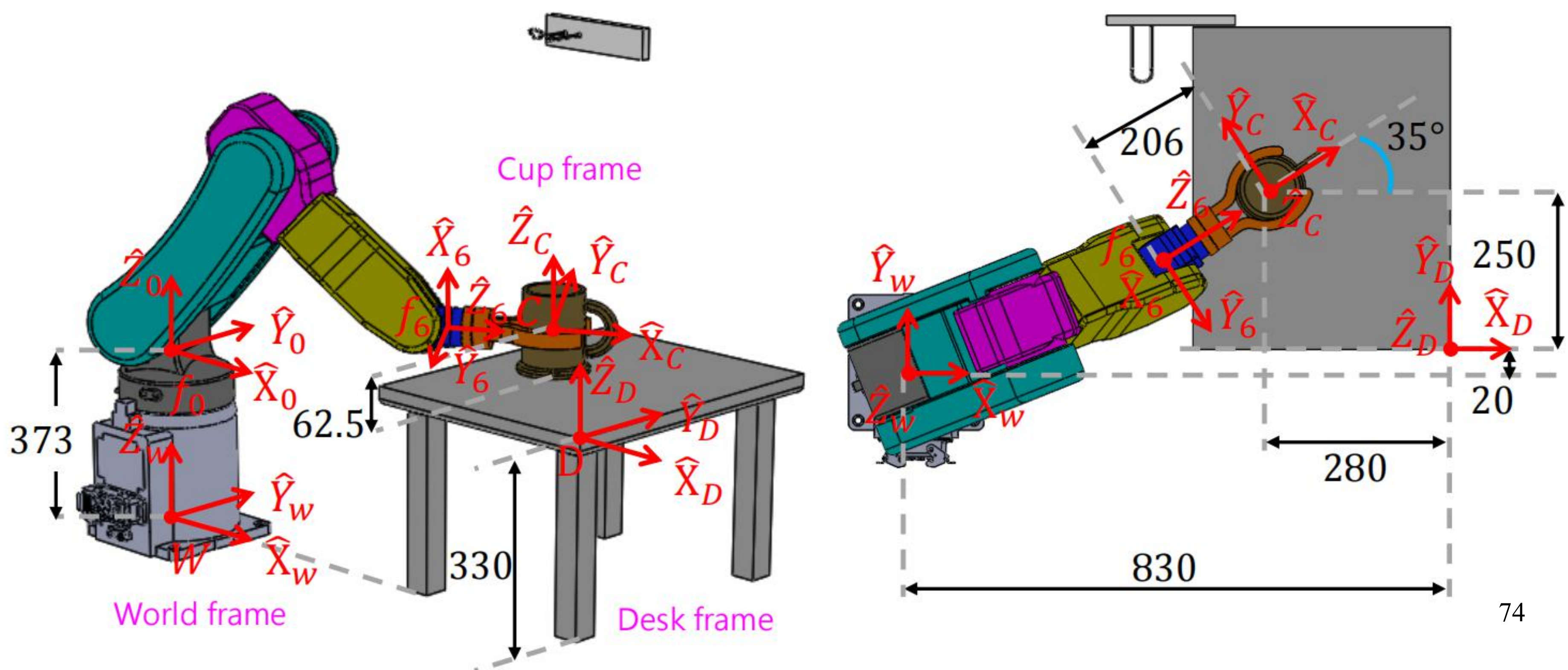
圖中顯示各軸為0°的狀態



i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0°	0	0	θ_1
2	-90°	$a_1=-30$	0	θ_2
3	0°	$a_2=340$	0	θ_3
4	-90°	$a_3=-40$	$d_4=338$	θ_4
5	90°	0	0	θ_5
6	-90°	0	0	θ_6

例：物件抓取任务

- Step 2: 找出 ${}^{W}_{C}T$，再進一步找出 ${}^{0}_{6}T$

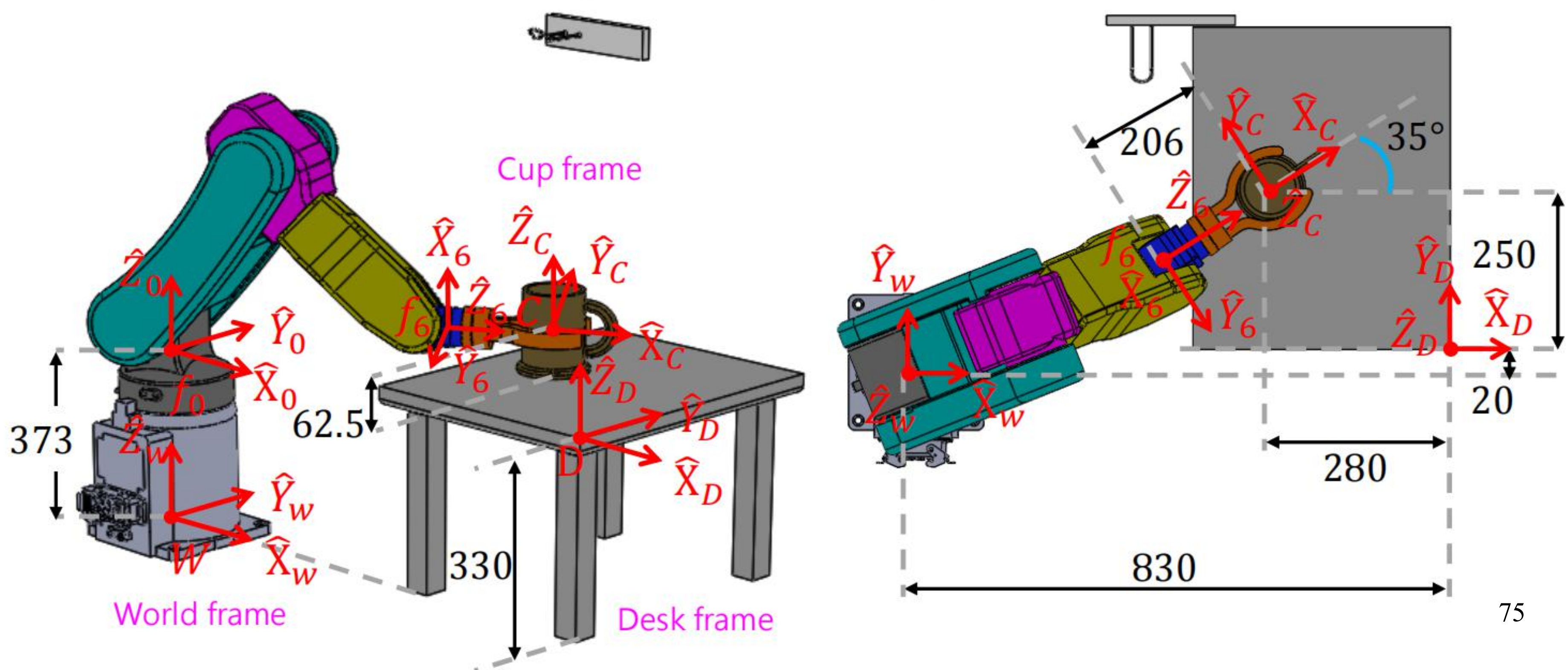


例：物件抓取任务

- Step 2: 找出 ${}^{W}_{C}T$，再進一步找出 ${}^{0}_{6}T$

$$
{}^{W}_{C}T = {}^{W}_{D}T\,{}^{D}_{C}T = \begin{bmatrix} 1 & 0 & 0 & 830 \\ 0 & 1 & 0 & 20 \\ 0 & 0 & 1 & 330 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos 35° & -\sin 35° & 0 & -280 \\ \sin 35° & \cos 35° & 0 & 250 \\ 0 & 0 & 1 & 62.5 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

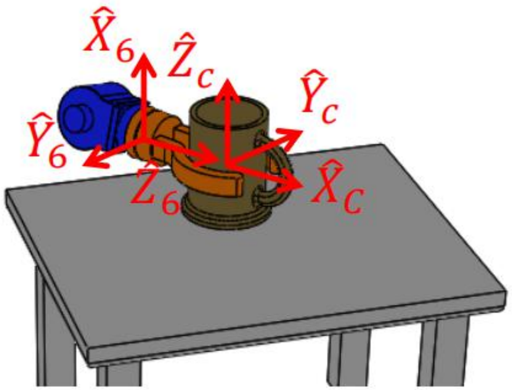
由「桌子相對於手臂」和「杯子相對於桌子」的相對關係推得

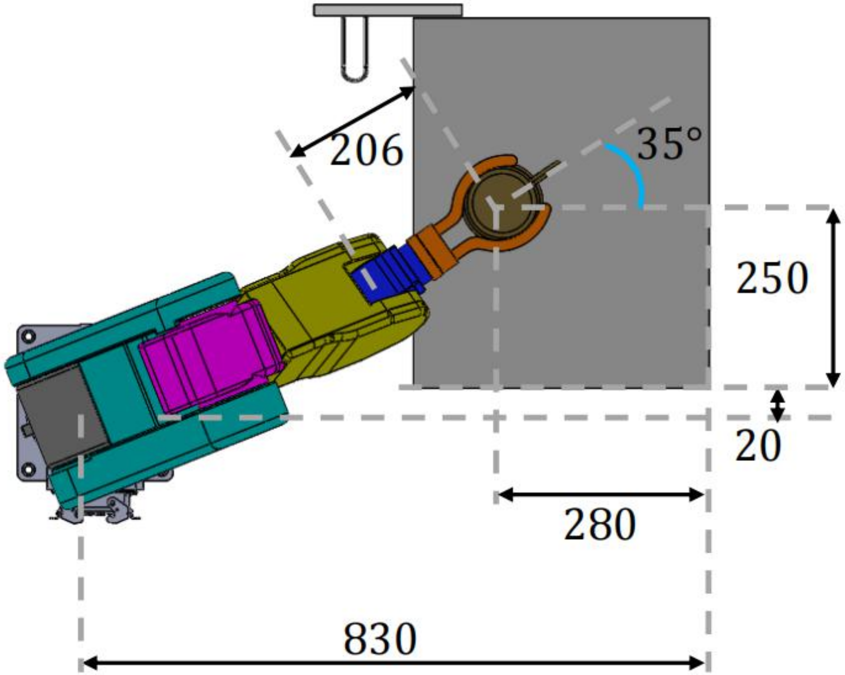


例：物件抓取任务

$$ {}^{W}_{C}T = {}^{W}_{0}T\,{}^{0}_{6}T\,{}^{6}_{C}T $$

$$ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 373 \\ 0 & 0 & 0 & 1 \end{bmatrix} {}^{0}_{6}T \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 206 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$





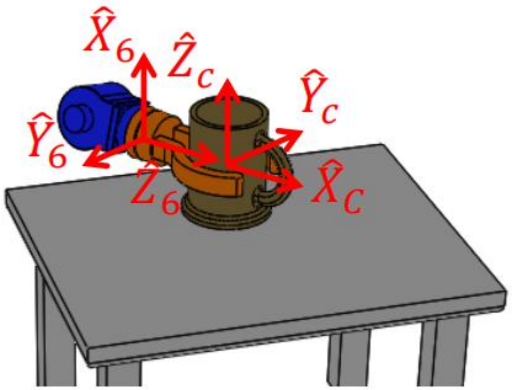
例：物件抓取任务

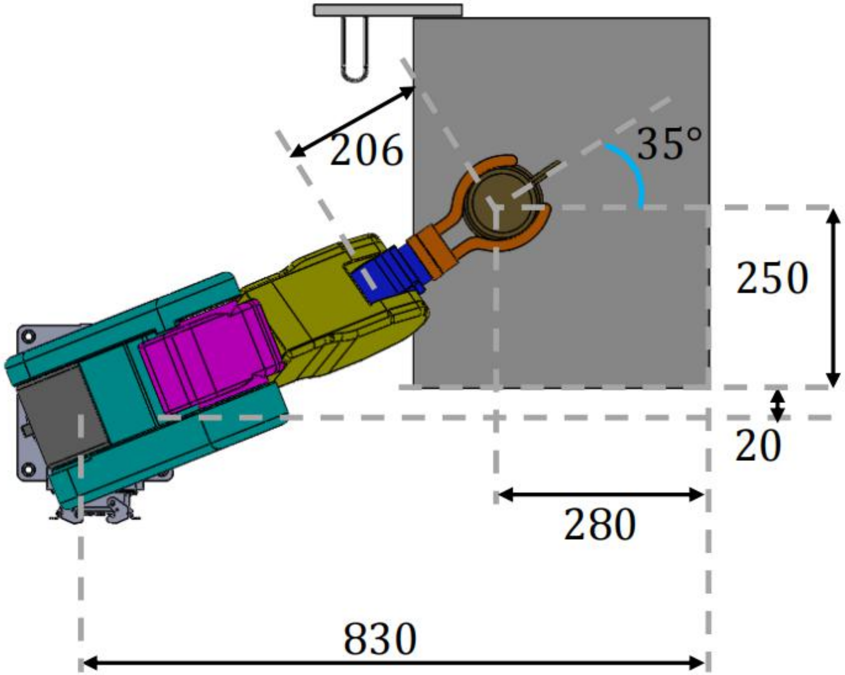
$$ {}^{W}_{C}T = {}^{W}_{0}T\,{}^{0}_{6}T\,{}^{6}_{C}T $$

$$ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 373 \\ 0 & 0 & 0 & 1 \end{bmatrix} {}^{0}_{6}T \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 206 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{0}_{6}T = {}^{W}_{0}T^{-1}\,{}^{W}_{C}T\,{}^{6}_{C}T^{-1} $$

$$ = \begin{bmatrix} 0 & 0.5736 & 0.8192 & 381.3 \\ 0 & -0.8192 & 0.5736 & 151.8 \\ 1 & 0 & 0 & 19.5 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$





例：物件抓取任务

$${}^{W}_{C}T = {}^{W}_{0}T\,{}^{0}_{6}T\,{}^{6}_{C}T$$

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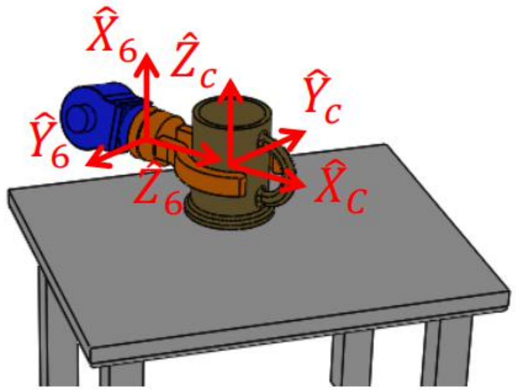
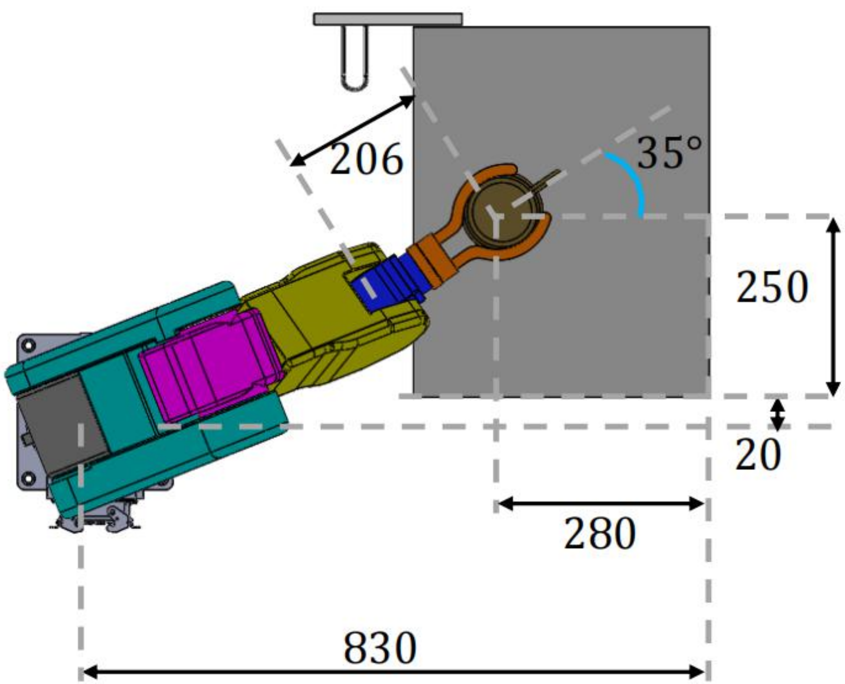
$${}^{0}_{6}R = \begin{bmatrix} 0 & 0.5736 & 0.8192 \\ 0 & -0.8192 & 0.5736 \\ 1 & 0 & 0 \end{bmatrix}$$

$${}^{0}P_{6\,ORG} = \begin{bmatrix} 381.3 \\ 151.8 \\ 19.5 \end{bmatrix}$$





例：物件抓取任务

- Step 3: 找出$\theta_1-\theta_6$
  - $\theta_1\ \theta_2\ \theta_3$角度求解

$$\begin{bmatrix} f_1(\theta_3) \\ f_2(\theta_3) \\ f_3(\theta_3) \\ 1 \end{bmatrix} = {}_3^2T\ {}^3P_{4\,ORG}$$

$$= \begin{bmatrix} c_3 & -s_3 & 0 & 340 \\ s_3 & c_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -40 \\ 338 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 340 - 338s_3 - 40c_3 \\ 338c_3 - 40s_3 \\ 0 \\ 1 \end{bmatrix}$$

例：物件抓取任务

- Step 3: 找出$\theta_1 - \theta_6$
  - $\theta_1\ \theta_2\ \theta_3$角度求解

$$\begin{bmatrix} f_1(\theta_3) \\ f_2(\theta_3) \\ f_3(\theta_3) \\ 1 \end{bmatrix} = {}^{2}_{3}T\ {}^{3}P_{4\,ORG}$$

$$= \begin{bmatrix} c_3 & -s_3 & 0 & 340 \\ s_3 & c_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -40 \\ 338 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 340 - 338s_3 - 40c_3 \\ 338c_3 - 40s_3 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} g_1(\theta_2,\theta_3) \\ g_2(\theta_2,\theta_3) \\ g_3(\theta_2,\theta_3) \\ 1 \end{bmatrix} = {}^{1}_{2}T \begin{bmatrix} f_1(\theta_3) \\ f_2(\theta_3) \\ f_3(\theta_3) \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} c_2 & -s_2 & 0 & -30 \\ 0 & 0 & 1 & 0 \\ -s_2 & -c_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f_1(\theta_3) \\ f_2(\theta_3) \\ f_3(\theta_3) \\ 1 \end{bmatrix} = \begin{bmatrix} 340c_2 - 40c_{23} - 338s_{23} - 30 \\ 0 \\ 40s_{23} - 338c_{23} - 340s_2 \\ 1 \end{bmatrix}$$

例：物件抓取任务

$$r = (k_1 c_2 + k_2 s_2) 2a_1 + k_3 = ||P||^2 = 168813.18$$

$$z = (k_1 s_2 - k_2 c_2) s\alpha_1 + k_4 = 19.5$$

例：物件抓取任务

$$r = (k_1 c_2 + k_2 s_2)2a_1 + k_3 = ||P||^2 = 168813.18$$
$$z = (k_1 s_2 - k_2 c_2)s\alpha_1 + k_4 = 19.5$$

計算θ_1 θ_2 θ_3角度

$$\frac{(r-k_3)^2}{4a_1^2} + \frac{(z-k_4)^2}{s^2\alpha_1} = k_1^2 + k_2^2 \qquad \Rightarrow solve\ \theta_3 = 2.5°$$

例：物件抓取任务

$$r = (k_1 c_2 + k_2 s_2) 2a_1 + k_3 = ||P||^2 = 168813.18$$
$$z = (k_1 s_2 - k_2 c_2) s\alpha_1 + k_4 = 19.5$$

計算θ_1 θ_2 θ_3角度

$$\frac{(r-k_3)^2}{4a_1^2} + \frac{(z-k_4)^2}{s^2\alpha_1} = k_1^2 + k_2^2 \qquad \Rightarrow solve\ \theta_3 = 2.5°$$

$$r = (k_1 c_2 + k_2 s_2) 2a_1 + k_3 \qquad \Rightarrow solve\ \theta_2 = -52.2°$$

$$x = c_1 g_1(\theta_2, \theta_3) - s_1 g_2(\theta_2, \theta_3) \qquad \Rightarrow solve\ \theta_1 = 21.8°$$

例：物件抓取任务

- $\theta_4\ \theta_5\ \theta_6$ 角度求解

$$ {}^{0}_{3}\mathrm{R}=\begin{bmatrix}0.6006 & 0.7082 & -0.3710\\ 0.24 & 0.2830 & 0.9286\\ 0.7627 & -0.6468 & 0\end{bmatrix} $$

$$ {}^{3}_{6}R={}^{0}_{3}R^{-1}\,{}^{0}_{6}R=\begin{bmatrix}0.7627 & 0.1477 & 0.6297\\ -0.6468 & 0.1744 & 0.7424\\ 0 & -0.9735 & 0.2286\end{bmatrix} $$

使用Z-Y-Z Euler angle求得剩下的joint angles

$\theta_4=-20°$　　$\theta_5=-42°$　　$\theta_6=15°$

谢 谢！